Determination of the orbit spaces of non-coregular compact linear groups with one relation among the basic polynomial invariants in the $\hat{P}$-matrix approach*

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Abstract

Invariant functions under the transformations of a compact linear group G acting in $I\!R^n$ can be expressed in terms of functions defined in the orbit space of G, i.e. as functions of a finite set of basic invariant polynomials $p(x) = (p_1(x), \ldots, p_q(x))$, $x \in I\!R^n$, which form an integrity basis (IB) for $(G, I\!R^n)$.

We develop a method to determine the isotropy classes of the orbit spaces of all the real linear groups whose IBs satisfy only one independent relation. The effectiveness of the method is tested for IB's fformed 3 (independent) basic invariants.

The result is obtained through the computation of a metric matrix $\widehat{P}(p)$, which is defined only in terms of the scalar products between the gradients $\partial p_1(x), \ldots \partial p_q(x)$, and whose domain of semi-positivity is known to realize the orbit space $I\!R^n/G$ of G as a semi-algebraic variety in the space $I\!R^q$ spanned by the variables $p_1, \ldots, p_q$.

After a short review of the approach that recently enabled to solve the analogous problem for coregular groups with less than 5 basic invariants, we determine the matrices $\widehat{P}(p)$ from the solutions of a universal differential equation (*master equation*), which satisfy new convenient additional conditions, which fit for the non-coregular case. The master equation involves as free parameters only the degrees d_a of the $p_a(x)$'s. This approach bypasses the actual impossibility of explicitely determining a set of basic polynomial invariants for each group.

Our results may be relevant in physical contexts where the study of covariant or invariant functions is important, like in the determination of patterns of spontaneous symmetry breaking in quantum field theory, in the analysis of phase spaces and structural phase transitions (Landau's theory), in covariant bifurcation theory, in crystal field theory and so on.

Keywords: Geometric invariant theory, Linear group actions, Orbit spaces, Non-coregular algebraic linear groups, Spontaneous symmetry breaking

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1 Introduction

Invariant functions under the transformations of a compact linear group (hereafter abbreviated in CLG) G, acting in an Euclidean space $\mathbb{R}^n$, play an important role in physics, and the determination of their properties is often a basic problem to solve in many physical contexts, such as the determination of patterns of spontaneous symmetry breaking, the analysis of phase spaces and structural phase transitions (Landau's theory), covariant bifurcation theory, crystal field theory and so on.

A G-invariant function $f(x)$, $x \in \mathbb{R}^n$, takes on constant values along each orbit of G, thus, if one has to analyze its properties, it is certainly more economical, and generally more effective, to think of it as a function defined in the orbit space $\mathbb{R}^n/G$ of the action of G in $\mathbb{R}^n$. In this way, it is possible to take fully into account the invariance properties of $f(x)$, while maintaining its regularity properties, but avoiding the troubles that could be met, for instance in the determination of the minima, owing to its degeneracy along the G-orbits.

This approach to the study of the properties of a G-invariant function, obviously requires a detailed knowledge of the structure of the orbit space $\mathbb{R}^n/G$, which can be obtained as follows, using the methods of invariant theory [1]. Let $\{p_1(x), \ldots, p_q(x)\}$ be a minimal integrity basis (hereafter abbreviated in MIB) of the ring $\mathbb{R}^n[x]^G$ of polynomial invariants of G. The MIB defines an *orbit map* $x \mapsto (p_1(x), \ldots, p_q(x)) \equiv p(x)$, mapping $\mathbb{R}^n$ onto a semi-algebraic subset, $p(\mathbb{R}^n)$, of $\mathbb{R}^q$, which provides a diffeomorphic image of $\mathbb{R}^n/G$ (see, for instance, [1], [2] or [3] and references therein). If $f(x)$ is a G-invariant polynomial or C^∞-function it can be written in terms of a polynomial or, respectively, C^∞-function $\hat{f}(p)$, in the form $f(x) = \hat{f}(p(x))$ [4, 5, 6]. The function $\hat{f}(p)$ has the same range as $f(x)$, but its domain is a faithful image of $\mathbb{R}^n/G$.

The price to pay in the orbit space approach to the analysis of a G-invariant function is essentially twofold:

1. MIB's are sometimes difficult to determine.

2. The domain of the associated function $\hat{f}(p)$ is not the whole Euclidean space $\mathbb{R}^q$, but reduces to the semi-algebraic subset $p(\mathbb{R}^n)$ [7, 8], not trivial to determine.

As for the problem stated under item 2, the following simple solution has been proposed. It has been shown [7, 8, 9] that, if the relations among the elements of the MIB's are known, the polynomial equalities and inequalities defining $p(\mathbb{R}^n)$ and its strata can be determined from the rank and positivity properties of a matrix $\widehat{P}(p)$, defined only in terms of the gradients of the elements of a MIB. When there are no relations among the elements of the MIB's, that is for compact *coregular* linear groups, the matrices $\widehat{P}(p)$ play the role of inverse metric matrices, and the isomorphism classes of the orbit spaces of all these groups can be classified in terms of equivalence classes of matrices $\widehat{P}(p)$.

A way to obtain the matrices $\widehat{P}(p)$ generated by CLG's, bypassing the actually insoluble problem of determining a MIB for each group, has been suggested in [10]. The idea is to use an axiomatic approach, that is to forget altogether the original definition of the matrices $\widehat{P}(p)$ and to characterize them as far as possible through their structural properties (*allowable* $\widehat{P}$-matrices). The possibility of actually computing them is favored by the identification and convenient formalization of a set of sufficiently strong, but handy, analytic conditions, shared by all $\widehat{P}(p)$'s. The less immediately evident of these conditions have been translated into a set of differential equations (*boundary conditions*) involving the elements of the matrices $\widehat{P}(p)$.

In the case of coregular groups, it has been proved that, in the case of less than 5 basic invariants [10, 11, 12], the allowable $\widehat{P}$-matrices can be determined from the solution of a unique universal differential equation (that here we shall call *master equation*), satisfying convenient initial conditions. The master equation involves as free parameters only the degrees d_a of the basic polynomial invariants. The conditions defining allowable $\widehat{P}$-matrices for *coregular* CLG's turn out to be so restrictive that, for each choice of a set of degrees $\{d_1, \ldots, d_q\}$, they turn out to select (at least for $q \leq 4$) only a finite number of equivalence classes of matrices $\widehat{P}(p)$. Thus, even if the results obtained along these lines are still strongly incomplete, one may reasonably hope to be able to obtain in the future a classification of the orbit spaces of all compact coregular linear groups, even if the classification of these groups is not yet complete and/or the explicit form of the elements of the corresponding MIB's is not known.

In this paper, we shall shortly review the geometry of linear group actions and describe how the invariant theory may successfully be applied to determine the stratification of the orbit space of a compact linear group. Then, we shall turn our attention on the axiomatic approach for the determination of the $\widehat{P}$-matrices of CLGs. We shall focus on new developments concerning the $\widehat{P}$-matrices generated by groups in the class $\mathcal{T}(q, q-1)$ of non-coregular groups with only one independent relation among the q elements of their MIB's. We shall show that the relation among the elements of a MIB is determined by one of the irreducible factors of the determinant of $\widehat{P}(p)$, which, along with $\widehat{P}(p)$, has to satisfy the master equation and some additional conditions, part of which can be put in the form of a subsidiary differential equation (*second order boundary conditions*). The effectiveness of these conditions is tested in the simplest case of three basic invariants, by determining all the solutions and selecting those which lead to allowable $\widehat{P}$-matrices.

In fact, the conclusion of our analysis is that, leaving aside the trivial cases in which at least one of the invariants $p_1(x), p_2(x), p_3(x)$ is linear,

- There is only one monoparametric discrete family of allowable non-equivalent $\widehat{P}$-matrices $\widehat{P}^{(k)}(p)$, $\mathbb{N} \ni k \geq 2$, whose elements may be generated by groups in $\mathcal{T}(3,2)$. The degrees of the p_a's are $d_1 = d_2 = k \geq 2$, $d_3 = 2$ and, with a convenient choice of the p_a's, the basic relation can be written in the form $\widehat{F}^{(k)}(p) = p_1^2 + p_2^2 - p_3^k$.
- Every allowable $\widehat{P}$-matrix of the family is generated by at least a group $G \in \mathcal{T}(3,2)$.
- If the action of the groups is restricted to the unit sphere $S^{(n-1)}$ of $\mathbb{R}^n$ (which is not

essentially restrictive for what concerns the characterization of the orbit space), all the orbit spaces $S^{(n-1)}/G$, $G \in \mathcal{T}(3,2)$ turn out to be isomorphic.

We shall present the matter in the following order. In Section 2 we shall recall some known results concerning the characterization of orbit spaces (see, for instance, [13, 3, 14, 8]) in the $\widehat{P}$-matrix approach. In Section 3 we recall the derivation of the boundary conditions, of the master relation and of the additional conditions that must be satisfied in the case of coregular CLG's. Suitable conditions are subsequently derived for non-coregular groups, with particular attention to the case in which there is only one basic relation: this is the original part of the paper. In §4 we formalize our approach to the determination of the $\widehat{P}$-matrices that could virtually be associated to actual non-coregular CLG's with one independent relation among the basic invariants; the boundary conditions and the master relation are presented as equations and the notions of proper and allowable solutions of the master equation are defined. In §5 we compute first all the proper and allowable solutions of the master equation and, after a further selection, we arrive at the determination of the equivalence classes of the $\widehat{P}$-matrices of all the non-coregular CLG's of class $\mathcal{T}(3,2)$. The last part of the section is devoted to the problem of the correspondence between allowable $\widehat{P}$-matrices and generating CLG's.

Our conclusions agree with a result about locally smooth actions on manifolds with orbits of codimension ≤ 2 [14, Th. IV 8.2, p. 206]. According to it, if the action of the group is restricted to the unit sphere $S^{(2)}$ of $\mathbb{R}^3$, then either $S^{(2)}/G$ is diffeomorphic to the unit interval $[0,1]$, or $S^{(2)}/G$ is diffeomorphic to the 1-sphere $S^{(1)}$. The former case refers to coregular groups of class $\mathcal{T}(2,2)$, the latter, both to non-coregular groups of class $\mathcal{T}(3,2)$ (whose orbit space is isomorphic to some $(\mathbb{Z}_m, \mathbb{R}^2)$, $m > 1$, where $\mathbb{Z}_m \subseteq \mathrm{SO}(2,\mathbb{R})$ acts on $\mathbb{R}^2$), and to coregular groups of class $\mathcal{T}(3,3)$ (e.g. the linear groups $\mathrm{SO}(n,\mathbb{R})$ acting in $\mathbb{R}^n \oplus \mathbb{R}^n$ for $n \geq 3$).

The fact that coregular and non-coregular groups may share orbit spaces belonging to the same isomorphism class is an intriguing fact that suggested us to treat the axiomatic $\widehat{P}$-matrix approach stressing, whenever possible, analogies and differences between those two cases.

It is worth noting that the effectiveness of our method does not depend on the dimension of the real vector space upon which G acts, so it may be applied, in principle, to determine the orbit spaces of all non-coregular groups with only one independent relation among the basic polynomial invariants.

2 An overview of the geometry of linear group actions

In this section, we shall first define most of our notations and recall, without proofs, some results concerning invariant theory and the geometry of orbit spaces of CLG's, then we shall introduce the first definitions and the basic tools for our subsequent analysis. For the unreferenced results see for instance [14, 3].

2.1 Orbits, strata and orbit spaces

For our purposes, it will not be restrictive to assume that G is a matrix subgroup of $O_n(I\!R)$ acting linearly in the Euclidean space $I\!R^n$.

We shall denote by $x = (x_1, \ldots, x_n)$ a point of $I\!R^n$. The group G acts in $I\!R^n$ in the following way:

$$x'_i = (g \cdot x)_i = \sum_1^n {}_j g_{ij}\, x_j\,, \qquad x \in I\!R^n,\ g \in G\,. \tag{1}$$

The G-orbit Ω_x through $x \in I\!R^n$ and the *isotropy subgroup* G_x of G at $x \in I\!R^n$ are defined by the following relations:

$$\Omega_x = \{g \cdot x \mid g \in G\}, \qquad G_x = \{g \in G \mid g \cdot x = x\}\,. \tag{2}$$

For all $x \in I\!R^n$, the isotropy subgroup G_x at x is a Lie group, which is not necessarily connected even if G is. If G is continuous, the Lie algebra $\mathcal{G}_x$ of G_x is formed by the elements of the Lie algebra $\mathcal{G}$ of G annihilating x; moreover, the G-orbits are smooth, closed and compact submanifolds of $I\!R^n$. They are connected if the group G is.

For G continuous, a tangent space $T_x(\Omega)$ to an orbit Ω can be defined at all $x \in \Omega$. It is formed by the tangent vectors at x to regular curves through x, lying in Ω. Therefore $T_x(\Omega)$, as a vector space, may be identified with $\{a \cdot x, a \in \mathcal{G}\}$, which is isomorphic to the Lie algebra quotient $\mathcal{G}/\mathcal{G}_x$. The normal space N_x to the orbit Ω through x is the orthogonal complement in $I\!R^n$ to $T_x(\Omega)$. It may be decomposed into the direct sum $N_x^{(0)} \oplus N_x^{(1)}$, where $N_x^{(0)}$ denotes the orthogonal invariant space to Ω at x, formed by all the vectors of $N_x(\Omega)$ which are invariant under G_x. Since it can be prooved that each component space in the decomposition $I\!R^n = T_x(\Omega) \oplus N_x^{(0)} \oplus N_x^{(1)}$ is globally G_x-invariant, the representation $(G_x, I\!R^n)$ induced by G in $I\!R^n$ turns out to be completely reducible.

If G is discrete, such as the finite subgroups of $\mathrm{O}(n, I\!R)$, it may be thought of as a compact Lie group with trivial connected component of the unit. The orbit Ω_x is a finite set formed by $\|G\|/\|G_x\|$ points. Therefore, the tangent space at each point $x \in I\!R^n$ reduces to the null vector of $I\!R^n$ and the normal space is the entire $I\!R^n$.

In any case, the invariance of the Euclidean norm under orthogonal transformations assures that the G–orbit through x is contained in the sphere of radius x, centered in the origin of $I\!R^n$, while the linearity of the action of G in $I\!R^n$ implies

$$G_x = G_{\lambda x},\ \forall \lambda \in I\!R_*\,. \tag{3}$$

The isotropy subgroup of G at the origin of $I\!R^n$ coincides with G. The isotropy subgroups of G, at points lying on the same orbit Ω_x are conjugate subgroups in G:

$$G_{g \cdot x} = g G_x g^{-1},\ \forall g \in G\,. \tag{4}$$

The class of all the subgroups of G conjugate to G_x in G will be said to be the *orbit type of* Ω_x and of its points; the orbit type specifies the symmetry properties of Ω_x under transformations induced by elements of G.

The points $x \in I\!R^n$ (or, equivalently, the orbits of G) sharing the same orbit type form an *isotropy type stratum of the action of* G *in* $I\!R^n$, hereafter called simply a *stratum of* $I\!R^n$. All the connected components of a stratum can be shown to be smooth iso-dimensional sub-manifolds of $I\!R^n$.

Since any sufficiently small displacement from a point x in the direction orthogonal to Ω_x does not change the symmetry properties of x iff that direction belongs to $N_x^{(0)}$, it is possible to prove that the following identity holds fot the tangent space $T_x(\Sigma)$ to the stratum in $I\!R^n$:

$$T_x(\Sigma) = T_x(\Omega) \oplus N_x^{(0)} \ . \tag{5}$$

The *orbit space* of the action of G in $I\!R^n$ is defined as the quotient space $I\!R^n/G$ (obtained through the equivalence relation between points belonging to the same orbit) endowed with the quotient topology and differentiable structure. We shall denote by π the canonical projection $I\!R^n \to I\!R^n/G$. Whole orbits of G are mapped by π into single points of $I\!R^n/G$. Any function f defined on $I\!R^n/G$ is differentiable iff $f \circ \pi$ is differentiable on $I\!R^n$. The image through π of a stratum of $I\!R^n$ will be called an (*isotropy type*) *stratum of* $I\!R^n/G$; all its connected components turn out to be smooth iso-dimensional manifolds.

Almost all the points of $I\!R^n/G$ belong to a unique stratum Σ_p, the *principal stratum*, which is a connected open dense subset of $I\!R^n/G$. The boundary $\overline{\Sigma_p}\backslash\Sigma_p$ of Σ_p is the union of disjoint *singular* strata. All the strata lying on the boundary $\overline{\Sigma}\backslash\Sigma$ of a stratum Σ of $I\!R^n/G$ are open in $\overline{\Sigma}\backslash\Sigma$.

The following partial ordering can be introduced in the set of all the orbit types: $[H] < [K]$ if H is conjugate to a subgroup of K. The orbit type $[H]$ of a stratum Σ is contained in the orbit types $[H_b]$ of all the strata Σ_b lying in its boundary; therefore, more peripheral strata of $I\!R^n/G$ are formed by orbits with higher symmetry under G transformations. The number of distinct orbit types of G is finite and there is a unique minimum orbit type, the *principal orbit type*, corresponding to the principal stratum; there is also a unique maximum orbit type $[G]$, corresponding to the image through π of the set of points of $I\!R^n$, which are invariant under G; this set contains at least the origin of $I\!R^n$.

Since any orbit Ω is mapped by π into a single point of the orbit space, (5) implies that the tangent space $T_\Omega(\hat{\Sigma})$ to a stratum $\hat{\Sigma}$ of $I\!R^n/G$ is isomorphic to the normal invariant space $N_x^{(0)}$, where x is any point belonging to the orbit $\Omega \in \hat{\Sigma}$. This fact has been exploited to construct a faithful image of the orbit space $I\!R^n/G$ in a Euclidean space[7, 8]. Before reviewing this result, we shall recall a few basics of the geometric approach to invariant theory.

A function $f(x)$ is said to be G-invariant if

$$f(g \cdot x) \ = \ f(x), \ \forall x \in I\!R^n, \ g \in G. \tag{6}$$

The set of all real, G-invariant, polynomial functions of x forms a ring $I\!R[x]^G$, that admits a finite integrity basis [4, 5]. Therefore, there exists a finite collection of invariant polynomials $p(x) = (p_1(x), p_2(x), \ldots, p_q(x))$ such that any element $F \in I\!R[x]^G$ can be expressed as a polynomial function $\widehat{F}$ of $p(x)$:

$$F(x) = \widehat{F}(p(x)), \forall x \in I\!R^n . \tag{7}$$

The polynomial function $\widehat{F}(p)$, $p \in I\!R^q$, will be said to have weight w, if w is the degree of the polynomial $F(x) = \widehat{F}(p(x))$ and it will be said to be w-homogeneous if $F(x)$ is homogeneous. We shall denote the homogeneity degree of F by $w(F)$.

The elements of a basis of $I\!R[x]^G$ can be chosen to be homogeneous polynomials. The number q of elements of a minimal integrity basis and their homogeneity degrees d_i's are only determined by the group G.

To avoid trivial situations, in this paper we shall only consider linear groups with no fixed points, but for the origin of $I\!R^n$. In this case, the minimum degree of the elements of a minimal integrity basis is necessarily 2, and the following conventions can be adopted:

$$d_1 \geq d_2 \geq \dots d_q = 2\,; \qquad p_q(x) = \|x\|^2 = \sum_1^n {}_i x_i^2\,. \tag{8}$$

Hereafter, by a *minimal integrity basis* (abbreviated into *MIB*) we shall always mean a *minimal homogeneous integrity basis* of the ring of G-invariant polynomials, so chosen that the conventions of (8) hold true.

The orbits of a compact group G are separated by the elements of any MIB of G, i.e., at least one element of a MIB takes on different values on two distinct orbits. But it can be said more. Each MIB $\{p_1(x), \dots, p_q(x)\}$, defines an orbit map $p : I\!R^n \longrightarrow I\!R^q$, $x \mapsto (p_1(x), \dots, p_q(x))$, which maps all the points of $I\!R^n$ lying on an orbit of G into a single point of $I\!R^q$. The range $p(I\!R^n)$ of the orbit map p yields a faithful image of the orbit space of G, and the elements $\{p_1, \dots, p_q\}$ of the MIB, thought of as coordinates in the space $I\!R^q$, provide a smooth parametrization of the points of $I\!R^n/G$. In fact, it can be shown that every orbit map induces a diffeomorphism of $I\!R^n/G$ onto a semi-algebraic connected closed subset $\overline{\mathcal{S}}$ of $I\!R^q$:

$$\overline{\mathcal{S}} = p(I\!R^n) \simeq I\!R^n/G\,. \tag{9}$$

The analysis of the structure of $I\!R^n/G$ is easier if one confines his attention to the orbit space of the action of G on the unit sphere S^{n-1} of $I\!R^n$. This is not restrictive for the following reasons. Owing to the linearity of the action of G, the isotropy subgroups of G at points lying on the same straight line through the origin of $I\!R^n$ coincide; thus an essentially complete specification of the structure of $I\!R^n/G$ is obtained from the structure of S^{n-1}/G. Indeed, there is a bijection Φ mapping the set $\{\Sigma_1\}$ of strata of S^{n-1}/G onto the set $\{\Sigma\}$ of strata of $(I\!R^n\backslash\{0\})/G$, such that $\Sigma = \Phi(\Sigma_1)$ is homeomorphic to $\Sigma_1 \times I\!R_+$. Moreover, the orbit space S^{n-1}/G is compact and connected. The same is consequently true for its image under an orbit map p: the semi-algebraic set

$$\overline{\mathcal{S}}_1 = p(S^{n-1}) = \overline{\mathcal{S}} \cap \Pi, \qquad \Pi = \{p \in I\!R^q \mid p^q = 1\} \tag{10}$$

is compact and connected.

2.2 Coregular and non–coregular groups

A set $\{p_1(x), \ldots, p_q(x)\}$ of G-invariant polynomials will be said to be *regular* if its elements are algebraically, and therefore functionally, independent. The linear group G will be said to be *coregular* if its ring of invariant polynomials admits regular integrity bases.

If $\{p\}$ is a MIB of a coregular linear group, the polynomial function $\widehat{F}(p)$ appearing in Eq. (7) is uniquely determined and, when it is w-homogeneous it satisfies the following relation, which is an immediate consequence of Euler equation $\sum_1^n {}_i x_i \partial_i F(x) = w(F) F(x)$:

$$\sum_1^q {}_a d_a p_a \, \partial_a \widehat{F}(p) = w(F) \widehat{F}(p). \tag{11}$$

Let $\{\widehat{F}_A(p)\}_{1 \leq A \leq K}$ be a complete set of basic homogeneous relations among the elements of a non-regular set $\{p_1(x), \ldots, p_q(x)\}$ of G-invariant homogeneous polynomials. The polynomials $\widehat{F}_A(p)$ can be chosen to be w-homogeneous and irreducible on the complex numbers. The associated equations

$$\widehat{F}_A(p) \; = \; 0, \qquad A = 1, \ldots, K \tag{12}$$

define an irreducible algebraic variety in $I\!\!R^q$ (and in $\mathbb{C}^q$ for $p \in \mathbb{C}^q$), which will be called the *variety $\mathcal{Z}$ of the relations* among the elements of the set. The variety $\mathcal{Z}$ has a singularity in $p = 0$. In fact, for all A, $\widehat{F}_A(p)$ is a w-homogeneous polynomial which cannot be solved polynomially with respect to anyone of the basic invariants p_a. The absence of linear terms in any p_a implies:

$$\widehat{F}_A(0) \; = \; 0 \, ; \qquad \partial \widehat{F}_A(0) \; = \; 0, \qquad A = 1, \ldots, K \, . \tag{13}$$

For $k = \dim(\mathcal{Z})$, the couple (q, k) will define the *regularity type* (hereafter called r-type) of the set $\{p\}$. If $\{p\}$ is a MIB of a group G, the couple (q, k) will define the *r-type* of G. If G is coregular, there are no relations among the elements of its MIB's and $\mathcal{Z} = I\!\!R^q$.

It will be worthwhile to note that, if there are relations among the elements of an integrity basis, the polynomial function $\widehat{F}(p)$ appearing in (7) is uniquely determined on $\mathcal{Z}$, but, as a polynomial function on $I\!\!R^q$, it is only determined modulo a polynomial $\widehat{F}_0(p)$ vanishing identically on $\mathcal{Z}$. In any case, however, its weight is uniquely determined and it will always be possible to choose $\widehat{F}(p)$ so that it satisfies (11). In the following, by a *w-homogeneous polynomial in p* of weight w we shall always mean a polynomial satisfying (11).

2.3 The $\widehat{P}(p)$ matrix

Let $\{p(x)\} = \{p_1(x), \ldots, p_q(x)\}$ be a MIB for the group G and $d_1, \ldots, d_q$, the corresponding weights. We shall associate to $\{p(x)\}$ the following square matrix, whose elements are G-invariant polynomials in x, defined only in terms of the G-invariant Euclidean scalar products $\langle \, \cdot \, , \, \cdot \, \rangle$ between the gradients of the elements of the set $\{p(x)\}$:

$$P_{ab}(x) \;=\; \langle\; \partial p_a(x)\;,\; \partial p_a(x)\;\rangle \;=\; \sum_1^n{}_i \frac{\partial p_a(x)}{\partial x_i}\cdot\frac{\partial p_b(x)}{\partial x_i} \;=\; \widehat{P}_{ab}(p(x))\,, \tag{14}$$

where $a, b = 1, \ldots, q$. In the last member of (14), use has been made of Hilbert's theorem, in order to express $P_{ab}(x)$ in terms of polynomial functions $\widehat{P}_{ab}(p_1, \ldots, p_q)$. As already noted, these functions are uniquely determined only on $\mathcal{Z}$.

The following properties, which are common to all the matrices $\widehat{P}(p)$, are more or less immediate consequences of their definition and of the conventions we have adopted:

P1. *Symmetry, homogeneity and bounds on the last row and column:* The matrix $\widehat{P}(p)$ is a real $q \times q$ symmetric matrix, whose elements $\widehat{P}_{ab}(p)$ can be chosen to be w-homogeneous polynomials of weight

$$w(\widehat{P}_{ab}) \;=\; d_a + d_b - 2\,. \tag{15}$$

Owing to the definition $p_q(x) \;=\; \sum_1^q{}_i x_i^2$, the last row and column of every matrix $\widehat{P}(p)$ are determined by the degrees of the MIB:

$$\widehat{P}_{qa}(p) \;=\; \widehat{P}_{aq}(p) \;=\; 2d_a p_a, \qquad a = 1, 2, \ldots, q\,. \tag{16}$$

The orbit space $\overline{\mathcal{S}}$, is a connected semi-algebraic sub-variety of $I\!R^q$ and, like all semi-algebraic varieties [15], it presents a natural stratification in connected semi-algebraic sub-varieties ${\sigma_i}^{(\alpha)}$, called *primary strata*[1]. The set $\overline{\mathcal{S}}$, therefore, consists in a finite collection of disjoint and connected semi-algebraic submanifolds of $I\!R^q$, $\{{\sigma_i}^{(\alpha)}\}_{i,\,\alpha}$, such that $\overline{\mathcal{S}} = \bigcup_{i\,,\alpha} {\sigma_i}^{(\alpha)}$ and the boundary of each ${\sigma_i}^{(\alpha)}$ is vacuum, or the union of lower-dimensional ${\sigma_j}^{(\beta)}$'s, $(j > i)$. Each ${\sigma_i}^{(\alpha)}$ is open in its closure and is defined recursively on i (which distinguishes semi-algebraic sets of different dimensions) as the α-th connected component of the set of regular points of the semi-algebraic varieties $\mathcal{W}_i = \overline{\mathcal{S}} \setminus \bigcup_{0\,<j<\,i} {\sigma_j}^{(\beta)}\,,\;\; i = 1, 2\ldots.$

A characterization of the image $\overline{\mathcal{S}} \;=\; p(I\!R^n)$ of the orbit space of G as a semi-algebraic variety can be easily obtained through the matrix $\widehat{P}(p)$ associated to one of its MIB's.

Let us introduce the result that permits to get an advantage out of the analysis in §2.1 about the local properties of the action of $(G, I\!R^n)$. It was prooved in [16] that the normal invariant space $N_x^{(0)}$ through x coincides with the vector space generated by the set $\Delta = \{\partial p_l(x)\}_{1\leq l\leq q}$ made up of the gradients at x of the elements of a MIB. The $\widehat{P}(p)$ matrix is then defined from the grammian matrix associated to the set of vectors Δ, which are a basis for the tangent space to the stratum $\hat{\Sigma}$ containing an orbit Ω through $x \in I\!R^n$. Therefore, the following fundamental theorem [7, 8, 9] holds true:

[1]A simple example of a compact connected semi-algebraic variety of $I\!R^3$ is yielded by a polyhedron. Its interior points form its unique 3-dimensional primary stratum, while 2-,1- and 0-dimensional primary strata are formed, respectively, by the interior points of each face, by the interior points of each edge and by each vertex.

Theorem 2.1 *Let G a compact matrix subgroup of $O_n(I\!R)$, p the orbit map $I\!R^n \to I\!R^q$ defined by the homogeneous MIB $\{p_1(x), p_2(x), \ldots, p_q(x)\}$ and $\widehat{P}(p)$ the matrix defined in* (14)*. Then $\overline{\mathcal{S}} = p(I\!R^n)$ is the semi-algebraic subset of the variety $\mathcal{Z} \subseteq I\!R^q$ of the relations among the elements of the MIB where $\widehat{P}(p)$ is positive semi-definite. The set $\overline{\mathcal{S}}$ is connected. The k–dimensional primary strata of $\overline{\mathcal{S}}$ are the connected components of the set $\widehat{W}^{(k)} = \{p \in \mathcal{Z} \mid \widehat{P}(p) \geq 0, \mathrm{rank}(\widehat{P}(p)) = k\}$; they are the images of the connected components of the k–dimensional isotropy type strata of $I\!R^n/G$. In particular the interior $\mathcal{S}$ of $\overline{\mathcal{S}}$, where $\widehat{P}(p)$ has the maximum rank, is the image of the principal stratum and is connected.*

The theorem assures that the orbit space of a coregular group is completely determined by the positivity conditions of a $\widehat{P}$-matrix computed from any one of its MIB's. For non-coregular groups, also a complete set of relations among the p_a's has to be specified; this however can be obtained from rank conditions on the matrix $\widehat{P}(p)$.

We shall also need the following property of the $\widehat{P}$-matrices associated to MIB's of CLG's:

P2. *Tensor character:* If $\{p_1, \ldots, p_q\}$ is a MIB, the matrix elements of $\widehat{P}(p)$ transform as the components of a rank 2 contravariant tensor under MIB transformations that maintain the conventions fixed in (8) (these transformations will be hereafter called MIBT's). In fact, let $\{p(x)\}$ and $\{p'(x)\}$ be distinct MIB's; the $p'_a(x)$'s, being G–invariant polynomials, can be expressed as polynomial functions of the $p_a(x)$'s[2]:

$$\begin{aligned} {p'}_\alpha &= {p'}_\alpha(p), \qquad \alpha = 1, \ldots, q-1\,, \\ p'_q &= p_q\,, \end{aligned} \tag{17}$$

where each of the polynomial functions $p'_\alpha(p)$ depends only on the p_β's whose weights d_β are not greater than d'_α. Then,

$$\widehat{P}'(p'(p)) \;=\; J(p) \cdot \widehat{P}(p) \cdot J^T(p)\,, \tag{18}$$

where we have denoted by $J(p)$ the Jacobian matrix of the transformation:

$$J_{ab}(p) \;=\; \partial {p'}_a(p)/\partial p_b, \qquad a, b = 1, \ldots, q\,; \tag{19}$$

the matrix J turns out to be upper–block triangular, with constant elements in the diagonal blocks, so that the determinant of $\widehat{P}(p)$ is a relative invariant of the group of the MIBT's.

2.4 Classification of the orbit spaces of compact linear groups

Two $\widehat{P}$-matrices $\widehat{P}(p)$ and $\widehat{P}'(p')$, computed from different MIB's $\{p\}$ and $\{p'\}$ of the same CLG will be said to be *equivalent*; the semi-algebraic varieties $\overline{\mathcal{S}}$ and $\overline{\mathcal{S}'}$ defined by the positivity conditions imposed on $\widehat{P}(p)$ and $\widehat{P}'(p')$ respectively, are isomorphic (as semi-algebraic varieties) realizations of the orbit space $I\!R^n/G$.

[2]Since in our conventions the q-th element of any MIB is fixed, when defining a MIBT we shall always understand the condition $p'_q = p_q$.

As stressed in the introduction, we shall try to characterize the $\widehat{P}$-matrices through their structural properties. So we shall need to extend the notion of equivalence to matrices endowed with the formal properties of the $\widehat{P}$-matrices.

Let $\{p_1, \ldots, p_q\}$ and $\{p'_1, \ldots, p'_q\}$ be two sets of "weighted" indeterminates, sharing the same set of weights $\{d_1, \ldots, d_q\}$, satisfying the first of our conventions in (8). Two matrices $\widehat{P}(p)$ and $\widehat{P}'(p')$, satisfying the conditions stated under **P1** will be said to be *equivalent* if they are connected by a relation like (18), where $J(p)$ is the Jacobian matrix of the transformation $p' = p'(p)$ endowed with the same formal properties of a MIBT.

The semi-algebraic varieties where two equivalent matrices $\widehat{P}(p)$ and $\widehat{P}'(p')$ turn out to be positive semi-definite, are clearly isomorphic.

On the basis of what we have just said, the notions of MIBT's (17) and of equivalence of $\widehat{P}$-matrices (18) can be extended to the case of different groups G and G', under suitable conditions.

Definition 2.1 *Let $\{p_1, \ldots, p_q\}$ and $\{p'_1, \ldots, p'_q\}$ be MIB's respectively of the compact linear groups G and G', sharing the same degrees $d'_j = d_j$, $j = 1, \ldots, q$. The orbit spaces $I\!R^n/G$ and $I\!R^{n'}/G'$ will be said to be isomorphic if there exists a formal MIBT $p' = p'(p)$ such that:*

i) for every $\widehat{F}'(p') \in \mathcal{I}(\mathcal{Z}')$, the function $\widehat{F}(p) = \widehat{F}'(p'(p)) \in \mathcal{I}(\mathcal{Z})$;

ii) the $\widehat{P}$-matrices $\widehat{P}(p)$ and $\widehat{P}'(p')$, associated to $\{p\}$ and $\{p'\}$ are equivalent.

If G and G' have isomorphic orbit spaces, then the images of their orbit spaces $\overline{\mathcal{S}}$ and $\overline{\mathcal{S}'}$, associated with the MIB's $\{p\}$ and $\{p'\}$ are isomorphic semi-algebraic varieties:

$$\overline{\mathcal{S}'} = p'(\overline{\mathcal{S}})\,. \tag{20}$$

Thus, in particular, the classification of the isomorphism classes of the orbit spaces of the *coregular* CLG's rests on the determination of a representative for each class of equivalent $\widehat{P}(p)$ matrices. As already noted, the orbit space of a non–coregular group can be determined from the knowledge of the $\widehat{P}$-matrix associated to one of its MIB's only if a complete set of basic relations among the elements of the MIB is specified.

3 Characterizing the matrices $\widehat{P}(p)$

In this section we shall point out a set of additional conditions that should characterize, as far as possible, the $\widehat{P}$-matrices associated to CLG's.

3.1 Boundary conditions

It has been shown in [10, 13] that, besides the constraints listed in §2.3 under **P1**, every $\widehat{P}$-matrix has to satisfy some additional conditions, that can be put in the form of a set of differential relations, so that one can try to determine the associated $\widehat{P}$-matrices associated to CLG's as solutions of a system of differential equations. Let us briefly recall the derivation of these results.

Let us denote by σ a general primary stratum of $\overline{\mathcal{S}}$, and by $\mathcal{I}(\sigma)$ the ideal formed by all the polynomials in $p \in I\!R^q$ vanishing on σ. Every $\hat{f}(p) \in \mathcal{I}(\sigma)$ defines in $I\!R^n$ an invariant polynomial function $f(x) = \hat{f}(p(x))$, and

$$f(x) \, = \, 0, \qquad \forall x \in \Sigma_f = p^{-1}(\sigma)\,. \tag{21}$$

The gradient $\partial f(x)$ is obviously orthogonal to Σ_f at every $x \in \Sigma_f$, but, it must also be tangent to Σ_f since $f(x)$ is a G–invariant function [14, 16]. As a consequence, it has to vanish on Σ_f:

$$0 \, = \, \partial f(x) \, = \, \sum_1^q {}_b \partial_b \hat{f}(p)\, \partial p_b(x) \Big|_{p=p(x)}, \qquad \forall x \in \Sigma_f\,. \tag{22}$$

By taking the scalar product of (22) with $\partial p_a(x)$, we end up with the following *boundary conditions*:

$$\sum_1^q {}_b \widehat{P}_{ab}(p)\, \partial_b \hat{f}(p) \in \mathcal{I}(\sigma), \qquad \forall \hat{f} \in \mathcal{I}(\sigma) \text{ and } \forall \sigma \subseteq \overline{\mathcal{S}}\,. \tag{23}$$

Equation (23) can be re-proposed in the form of a differential relation involving only polynomial functions of p. According to the Hilbert basis theorem [17], the ideal $\mathcal{I}(\sigma)$ is finitely generated. Let $\{f^{(1)}(p), f^{(2)}(p), \ldots, f^{(m)}(p)\}$ be a w-homogeneous basis for $\mathcal{I}(\sigma)$, then (23) is equivalent to the following relations:

$$\sum_1^q {}_b \widehat{P}_{ab}(p)\, \partial_b a^{(r)}(p) \, = \, \sum_1^m {}_s \lambda_a^{(rs)}(p)\, a^{(s)}\,, \qquad a = 1, \ldots, q\,; \;\; r = 1, \ldots, m\,, \tag{24}$$

where the $\lambda^{(rs)}$'s are w-homogeneous polynomial functions of p of weight $(w(a^r) - w(a^s) + d_a - 2)$.

It is easy to realize that

P3. *The $a^{(r)}$ transform like relative invariants and the $\lambda^{(rs)}(p)$ like vector fields, under MIB transformations.*

In the particular case in which σ is a $(q-1)$-dimensional primary stratum, the ideal $\mathcal{I}(\sigma)$ has a unique *irreducible* generator, $a(p)$, and (24) reduces to the simpler form

$$\sum_1^q {}_b \widehat{P}_{ab}(p)\, \partial_b a(p) \, = \, \lambda_a(p)\, a(p)\,, \qquad a = 1, \ldots, q\,. \tag{25}$$

Equation (25) will be quoted as *master relation*[3].

There are only two types of $(q-1)$-dimensional strata:

[3] The present extension of the $\widehat{P}$-matrix approach to the non-coregular case has stimulated a refinement of the definition of the *canonical equation* with respect to the one appeared in [10, 11, 13].

1. Sub-principal strata of orbit spaces of coregular groups. This case will be discussed in § 3.2.

2. Principal strata of orbit spaces of groups of r-type $(q, q-1)$. This case will be discussed in § 3.3.

The structure of (25) has been analyzed in Ref. [11] where the results summarized below have been proved. We shall need them in the following.

i) $a(p)$ is a polynomial factor of $\det \widehat{P}(p)$; it will be called an *active* factor of $\det \widehat{P}(p)$.

ii) If, for a given $\widehat{P}(p)$, the couples $(a^{(i)}(p), \lambda^{(i)}(p))$, $i = 1, \ldots, K$, satisfy the master relation (25), then the couple $(A(p) = \prod_1^K{}_i a^{(i)}(p),\ \lambda(p) = \sum_1^K{}_i \lambda^{(i)}(p))$ satisfies the master relation.

iii) If, for a given MIB $\{p'\}$, the couple $(A'(p'), \lambda'(p'))$, satisfies the master relation (25), there exist particular MIB's, which we shall call *A-bases*, in which the vector $\lambda(p)$ reduces to the simple *canonical* form:

$$\lambda_a(p) = 2\delta_{aq} w(A)\,, \qquad a = 1, \ldots, q\,. \tag{26}$$

In an A-basis, the master relation assumes the following *canonical* form:

$$\sum_1^q{}_b \widehat{P}_{ab}(p)\partial_b A(p) \;=\; 2\delta_{aq} w(A) A(p), \qquad a = 1, \ldots, q\,. \tag{27}$$

Remark 3.1 The studies about structural phase transitions in the Landau approach have stimulated some authors [22, 23, 24] to examine the problem of the determination of a basis for covariant vector fields (or, in general, tensor fields) for the action of a (point) group G. Although the solution of this problem is not necessary to determine the minima of the Landau free energy, the role of the vector fields was analized in detail in the mathematical literature [2, 3]. In particular, it has been prooved that a smooth vector fields V on the image of the orbit space $\overline{\mathcal{S}}$ is tangent to a stratum σ iff it preserves the ideal $\mathcal{I}(\sigma)$ in $\mathcal{C}^\infty(\overline{\mathcal{S}}, I\!\!R)$ of the real valued smooth functions vanishing on σ.

In this context, the columns of the $\widehat{P}$-matrix may be viewed as the components of a strata preserving vector field.

3.2 Additional conditions for coregular groups

If $\{p\}$ is a regular MIB, there is a unique (generally reducible) generator, $A(p)$, of the ideal $\mathcal{I}(\mathcal{B})$, associated to the union $\mathcal{B}$ of all the $(q-1)$-dimensional strata of $\overline{\mathcal{S}}$[4] and it satisfies (27).

The following results have been proved in [11] to hold true in every A-basis:

[4] The closure of $\mathcal{B}$ forms the boundary of $\overline{\mathcal{S}}$.

i) The point $p^{(0)} = (0, \ldots, 0, 1)$ lies in the interior $\mathcal{S}$ of $\overline{\mathcal{S}}$; it is the image of a particular G-orbit lying on the unit sphere of $I\!R^n$.

ii) $A(p)$ is a factor of $\det \widehat{P}(p)$; it can be normalized at $p^{(0)}$:

$$A(p^{(0)}) = 1 \tag{28}$$

and its weight is bounded:

$$2d_1 \leq w(A) \leq w(\det \widehat{P}) = 2\sum_1^q {}_a d_a - 2q\,. \tag{29}$$

iii) The restriction $A(p)|_{p_q=1}$, of $A(p)$ to the hyperplane Π of $I\!R^q$, has a unique local non degenerate maximum lying at $p^{(0)}$; thus:

$$\partial_\alpha A(p)|_{p=p^{(0)}} = 0, \qquad \alpha = 1, \ldots, q-1\,. \tag{30}$$

iv) $\widehat{P}(p^{(0)})$ is block diagonal, each block being associated to a subset of p_a's sharing the same weight, and, in a subclass of A-bases (*standard A-bases*), it is diagonal:

$$\widehat{P}_{ab}(p^{(0)}) = d_a d_b \delta_{ab}, \qquad a, b = 1, \ldots, q. \tag{31}$$

3.3 Additional conditions for non-coregular groups

The $\widehat{P}$-matrices generated by non-coregular groups do not satisfy the set of "initial conditions" specified in the preceding subsection, but the presence of relations connecting the elements of any MIB gives rise to constraints that we shall try and formalize in a convenient way in this section. We shall first deal with general non-coregular groups; subsequently the results will be specialized to groups of r-type $(q, q-1)$.

3.3.1 Second order boundary conditions. The general case

Let us consider a compact non-coregular group G, whose orbit space, according to Theorem 2.1, is realized as a semi-algebraic subset $\overline{\mathcal{S}}$ of the variety $\mathcal{Z}$ of the relations.

We shall denote by $\mathcal{I}(\mathcal{Z})$ the ideal of the polynomial functions of p vanishing on $\mathcal{Z}$. Any polynomial $\widehat{F}(p) \in \mathcal{I}(\mathcal{Z})$ defines an identity in $I\!R^n$:

$$F(x) = \widehat{F}(p(x)) = 0\,, \tag{32}$$

which, after differentiating twice with respect to x_i and summing over i, gives rise to the following condition, valid $\forall x \in I\!R^n$:

$$\sum_1^n{}_i \left\{ \sum_1^q{}_{a,b} \left(\partial_a \partial_b \widehat{F}\right)(p(x))\, \partial_i p_b(x)\, \partial_i p_a(x) + \sum_1^q{}_a \left(\partial_a \widehat{F}\right)(p(x))\, \partial_i^2 p_a(x) \right\} \;=\; 0\,. \tag{33}$$

Since G is a matrix subgroup of $\mathrm{O}_n(I\!\!R)$, the n-dimensional Laplacian of any invariant polynomial function of x is a G-invariant polynomial. Thus Hilbert's theorem ensures the existence of a collection of polynomial functions $\hat{l}_a(p)$ $(a = 1, 2, \ldots, q)$ of $I\!\!R^q$ such that:

$$l_a(x) \;=\; \sum_1^n{}_i \partial_i^2 p_a(x) \;=\; \hat{l}_a(p(x))\,, \qquad a = 1, 2, \ldots, q\,. \tag{34}$$

From their very definition the $l_a(x)$'s are homogeneous polynomials with degrees $d_a - 2$; therefore the $\hat{l}_a(p)$ can be chosen to be w-homogeneous polynomials in p with weights

$$w(\hat{l}_a) = d_a - 2. \tag{35}$$

Since in our approach the explicit form of the polynomials $p_\alpha(x)$, $\alpha = 1, 2, \ldots, q-1$, is not specified, in the following the $\hat{l}_\alpha(p)$ have to be thought of as unknown $w-$homogeneous polynomials, while, owing to the convention $p_q = \sum_{j=1}^n x_j^2$:

$$l_q \;=\; 2n \;\geq 4 \tag{36}$$

and the inequality is a consequence of the fact that, in our assumptions, the dimension n of the vector space in which G acts must be ≥ 2.

The identity expressed in (33) induces, through the orbit map, the following polynomial relation, valid for all $p \in \overline{\mathcal{S}} \;=\; p(I\!\!R^n)$, and consequently, for all $p \in \mathcal{Z}$, as $p(I\!\!R^n)$ is a semi-algebraic subset of $\mathcal{Z}$ of the same dimension as $\mathcal{Z}$:

$$\sum_1^q{}_{a,b} \widehat{P}_{ab}(p)\; \partial_a \partial_b \widehat{F}(p) + \sum_1^q{}_a \hat{l}_a(p)\; \partial_a \widehat{F}(p) \;=\; 0\,, \qquad p \in \mathcal{Z}\,. \tag{37}$$

It is now evident that (37) may be considered as a sort of *second order boundary condition.*

Let us denote by $\{\widehat{F}_A(p)\}$, $1 \leq A \leq K$ a w-homogeneous minimal basis of the polynomial ideal $\mathcal{I}(\mathcal{Z})$, then (37) can be rewritten in the following equivalent form:

$$\sum_1^q{}_{a,b} \widehat{P}_{ab}(p)\; \partial_a \partial_b \widehat{F}(p) \;+ \sum_1^q{}_a \hat{l}_a(p)\;\; \partial_a \widehat{F}(p) \;=\; \sum_1^m{}_A \xi_A(p) \widehat{F}_A(p)\;, \tag{38}$$

where the $\xi_A(p)$'s are w-homogeneous polynomials.

The r.h.s. of (38) has to be a w-homogeneous polynomial of weight $(w(\widehat{F}) - 2)$ like the l.h.s.. As a consequence, in the particular case in which $\widehat{F}$ is a lowest weight element of the ideal $\mathcal{I}(\mathcal{Z})$, the second member of Eq. (38) must be zero. Thus we have proved the following proposition:

Proposition 3.1 *Let $\widehat{F}(p)$ be a lowest weight generator of the ideal $\mathcal{I}(\mathcal{Z})$ generated by the polynomial relations among the elements of a MIB. Then*

$$\mathrm{Trace}\left(\widehat{P}(p)\cdot \mathrm{He}\left(\widehat{F}(p)\right)\right)+\langle\ \hat{l}(p)\ ,\ \partial\widehat{F}(p)\ \rangle\ =\ 0\,, \tag{39}$$

where $\langle\ \cdot\ ,\ \cdot\ \rangle$ denotes the Euclidean scalar product in $\mathbb{R}^q$,

$$\left(\mathrm{He}\left(\widehat{F}(p)\right)\right)_{a,\,b}\ =\ \partial_a\partial_b\widehat{F}(p)\,,\qquad a,b=1,\ldots,q\,, \tag{40}$$

and the $\hat{l}_a$'s have been defined in (34).

3.3.2 Groups of r-type $(q, q-1)$

In the case of groups of r-type $(q, q-1)$, the ideal $\mathcal{I}(\mathcal{Z})$ of the relations among the elements of a MIB $\{p\}$ has a unique generator $\widehat{F}(p)$, so that the master relation is satisfied by $\widehat{F}(p)$:

$$\sum_1^q{}_b\widehat{P}_{ab}(p)\,\partial_b\widehat{F}(p)\ =\ \lambda_a(p)\,\widehat{F}(p) \tag{41}$$

and Eq. (39) holds also true.

The second order derivatives of $\widehat{F}(p)$ in (39) can be eliminated. In fact, by differentiating (41) with respect to p_a and summing over a, one obtains

$$\sum_1^q{}_{a,b}\widehat{P}_{ab}(p)\partial_a\partial_b\widehat{F}(p)\ =\ \sum_1^q{}_b\left(-\sum_1^q{}_a\partial_a\widehat{P}_{ab}(p)+\lambda_b(p)\right)\partial_b\widehat{F}(p)\,,$$

where use has been made of the fact that $\partial\lambda_a/\partial p_a=0$, since the weight of λ_a is d_a-2. After replacing in (39) and defining the w-homogeneous vector field $L_a(p)$ by

$$L_a(p)\ =\ \hat{l}_a(p)+\lambda_a(p)-\sum_1^q{}_b\partial_b\widehat{P}_{ab}(p)\,, \tag{42}$$

we obtain

$$\sum_1^q{}_aL_a(p)\partial_a\widehat{F}(p)\ =\ 0\,. \tag{43}$$

The weights of the components of $L(p)$ are the following:

$$w(L_a)\ =\ d_a-2. \tag{44}$$

In particular L_q is a constant that can be easily calculated, from (16), (26) and (36):

$$L_q=2\left(n+w(F)-\sum_1^q{}_ad_a\right)\ =\ 2D\,, \tag{45}$$

with

$$D \geq w(F) - \sum_{1}^{q-1}{}_{\alpha} d_\alpha \,, \tag{46}$$

where the lower bound is a consequence of the relations $d_q = 2$ and $n \geq 2$.

4 The master equation and allowable $\widehat{P}$-matrices

As already stressed, the orbit space of a CLG can be characterized through an associated $\widehat{P}$-matrix and the specification of the surface $\mathcal{Z}$, in case of non-coregularity. Our final aim is the determination and classification of the orbit spaces of the CLG's, avoiding to pass through an explicit determination of a MIB for each group.

For that reason, following the same approach proposed in [10], and developed in [11], for coregular groups, we shall now look at the boundary conditions from a different point of view. We shall forget altogether the group G and the space $I\!R^n$, and we shall think of $\{p_1, \ldots, p_q\}$ as a set of *weighted indeterminates*, with integer weights $d_1, \ldots, d_q$ satisfying the following conditions:

$$d_1 \geq d_2 \geq \ldots d_q \, = \, 2 \, . \tag{47}$$

We shall associate *formal* $\widehat{P}$-matrices satisfying condition **P1** in §2.3 to a weighted set $\{p\} = \{p_1, \ldots, p_q\}$.

Definition 4.1 *Let* $\{p\} = \{p_1, \ldots, p_q\}$ *be a set of weighted real variables of weights* $\{d\} = \{d_1, \ldots, d_q\}$ $(d_1 \geq \ldots \geq d_q = 2)$. *We shall say that* $\widehat{P}(p)$ *is a* (formal) $\widehat{P}$-matrix *associated with* $\{p\}$*, if it satisfies the following conditions:*

i) $\widehat{P}(p)$ *is a real, symmetric,* $q \times q$ *matrix.*

ii) The matrix elements $\widehat{P}_{ab}(p)$ *are* w*-homogeneous polynomials in* p *and their weights are* $w(\widehat{P}_{ab}) = d_a + d_b - 2, \; a, b = 1, \ldots, q$.

iii) $P_{qa}(p) = 2 d_a p_a, \; a = 1, \ldots, q$.

The equivalence of two formal $\widehat{P}$-matrices will be defined as in § 2.4.

The boundary relations will be considered as *equations* in which the polynomial functions involved play the role of unknown polynomial functions of p. With the above meaning for the symbols, Eq. (25) will be called the *master equation* and its canonical form (Eq. (27)) *the canonical equation.*

We shall be interested in solutions $\left(\widehat{P}(p), A(p)\right)$ of the canonical equation satisfying some additional conditions, which are certainly satisfied by all $\widehat{P}$-matrices originating from CLG's and associated to convenient MIB's.

Definition 4.2 *A* $\widehat{P}$*-matrix* $\widehat{P}(p)$ *associated to the weighted set of variables* $\{p\} = \{p_1, \ldots, p_q\}$ *of weights* $\{d\} = \{d_1, \ldots, d_q\}$ $(d_1 \geq \ldots \geq d_q = 2)$*, will be said to be* allowable of r-type (q, k) *if it satisfies the following conditions:*

i) There is a k-dimensional ($k \geq 1$) irreducible algebraic surface $\mathcal{Z}$ in $I\!R^q$ whose defining equations $\widehat{F}_A(p) = 0$ can be expressed in terms of irreducible w-homogeneous polynomials $\widehat{F}_A(p)$, such that $\partial\widehat{F}_A(0) = 0$. On $\mathcal{Z}$, $\operatorname{rank}(\widehat{P}(p)) \leq k$ and the set $\mathcal{R} = \{p \in \mathcal{Z} \,|\, \widehat{P}(p) \geq 0\,, \operatorname{rank}(\widehat{P}(p)) = k\}$ is k-dimensional and connected; the closure $\overline{\mathcal{R}}$ of $\mathcal{R}$ coincides with the set $\mathcal{R}^{(\geq)} = \{p \in \mathcal{Z} \,|\, \widehat{P}(p) \geq 0\}$.

ii) $\widehat{P}(p)$ satisfies the boundary conditions (23), for each primary stratum ${\sigma_i}^{(\alpha)}$ of $\overline{\mathcal{R}}$, and the second order boundary condition (37) for each $\widehat{F}(p) \in \mathcal{I}(\mathcal{Z})$.

In the following, the expression *allowable $\widehat{P}$-matrix* will be abbreviated in $A\widehat{P}M$.

Remark 4.1 The conditions under item *i)* in the Def. (4.2) imply immediately that for an $A\widehat{P}M$, $\operatorname{rank}(\widehat{P}(p)) < k$ on the boundary $\overline{\mathcal{R}} \setminus \mathcal{R}$ of $\mathcal{R}$.

Remark 4.2 In the first definition of $A\widehat{P}M$'s [11], a condition of compactness of the set $\Pi \cap \overline{\mathcal{R}}$ was included. From the proof of Theorem 7.1 in [11], it is easy to realize that the set $\Pi \cap \overline{\mathcal{R}}$ is compact for every formal $\widehat{P}$-matrix, as a consequence of its structure.

The $A\widehat{P}M$'s of r-type (q, q), that could be associated to coregular groups, have all been determined in [10, 11, 12] for $q \leq 4$, solving the canonical equation with the initial conditions specified in (28) and (31). We shall call these solutions (and the associated $\widehat{P}$-matrices), *proper solutions (proper $\widehat{P}$-matrices) of r-type (q, q).*

Analogously, in the following section we shall determine the $A\widehat{P}M$'s that could be associated to non-coregular groups of r-type $(q, q-1)$ from the solutions of the canonical equation which are proper of r-type $(q, q-1)$ in the sense specified by the following definition:

Definition 4.3 *Let $\widehat{P}(p)$ be a $\widehat{P}$-matrix associated to the weighted set of variables $\{p_1, \ldots, p_q\}$ and $\left(\widehat{P}(p), \widehat{F}(p)\right)$ be a solution of the canonical equation. The couple $\left(\widehat{P}(p), \widehat{F}(p)\right)$ will be said to be a* proper solution of r-type $(q, q-1)$ *if it satisfies the following conditions:*

i) $\widehat{F}(p)$ is irreducible on the complex field,

ii) $\partial\widehat{F}(0) = 0$,

iii) There are w-homogeneous polynomials $L_a(p)$ such that Eqs. (43) and (46) are satisfied.

A $\widehat{P}$-matrix obtained from a proper solution will be said to be a proper $\widehat{P}$-matrix (henceforth abbreviated in $P\widehat{P}M$).

In the following section for each proper solution of the canonical equation we shall determine the conditions under which the associated $\widehat{P}$-matrix is allowable. In order to determine the semi-positivity domain of $\widehat{P}(p)$, we shall make use of the following well known theorem and of a lemma which we shall prove below;

Theorem 4.1 *Let A be a real symmetric matrix. Then $A \geq 0$ if and only if $A_\alpha \geq 0$ for all α, where $\{A_\alpha\}$ is the set of determinants of principal (i.e., symmetric) minors of A.*

Lemma 4.1 *Let $(\widehat{P}(p), \widehat{F}(p))$ be a proper solution of r-type $(q, q-1)$ and $\mathcal{Z} = \{p \in I\!R^q \mid \widehat{F}(p) = 0\}$, then, with the same meaning of the symbols as in* Def. (4.2)*, the following two conditions are necessary and sufficient for $\widehat{P}(p)$ being an $A\widehat{P}M$ of r-type $(q, q-1)$:*

i) The semi-algebraic set $\mathcal{R}_1 = \Pi \cap \mathcal{R}$ is $(q-2)$-dimensional and connected and its closure $\overline{\mathcal{R}_1}$ coincides with the semi-algebraic set $\mathcal{R}_1^{\geq} = \Pi \cap \mathcal{R}^{\geq}$.

ii) $\widehat{P}(p)$ satisfies the boundary conditions (23) *at every singular primary stratum of $\mathcal{R}^{(\geq)}$.*

Proof: Since in the statement of the lemma item *ii)* essentially coincides with item *ii)* in Def. (4.2), it will be sufficient to note the following facts.

a) From $\widehat{P}_{qq}(p) = 4p_q$ and Theorem 4.1 it follows that $\widehat{P}(p) \geq 0$ only for $p_q \geq 0$.

b) Owing to w-homogeneity properties of $\widehat{F}(p)$, the point $p = 0$ belongs to $\mathcal{Z}$ and $\widehat{P}(0) = 0$. The origin of $I\!R^q$ is the only point where $\operatorname{rank}(\widehat{P}(p)) = 0$; moreover, if $p \in \mathcal{Z}$, then $\left(s^{d_1}p_1, \ldots, s^{d_{q-1}}p_{q-1}, s^2 p_q\right) \in \mathcal{Z}$ for all $s \in I\!R$.

c) Let us denote by $r : I\!R^q \longrightarrow \Pi$, the map such that $p \equiv (p_1, \ldots, p_{q-1}, p_q) \mapsto r(p) \equiv \left(\frac{p_1}{{p_q}^{d_1/2}}, \frac{p_2}{{p_q}^{d_2/2}}, \ldots, \frac{p_{q-1}}{{p_q}^{d_{q-1}/2}}, 1\right)$, which is well defined $\forall p_q > 0$.

Then, the w-homogeneity properties of $\widehat{P}(p)$ assure that for $p_q > 0$:

$$p_q \widehat{P}(p) \;=\; T\,\widehat{P}(r(p))\,T\,, \tag{48}$$

where $T = \operatorname{diag}({p_q}^{d_1/2}, {p_q}^{d_2/2}, \ldots, {p_q}^{d_{q-1}/2}, p_q)$. As a consequence, $\widehat{P}(p) \geq 0$ if and only if $\widehat{P}(r(p)) \geq 0$, and, using also the w-homogeneity properties of $\widehat{F}(p)$ mentioned under item *b)*, the following relations hold true:

$$\mathcal{R} \;=\; \left\{\left(s^{d_1}r_1(p), \ldots, s^{d_{q-1}}r_{q-1}(p), s^2\right) \;\mid\; r(p) \in \mathcal{R}_1\,,\, s > 0\right\}\,, \tag{49}$$

$$\mathcal{R}^{(\geq)} \;=\; \left\{\left(s^{d_1}r_1(p), \ldots, s^{d_{q-1}}r_{q-1}(p), s^2\right) \;\mid\; r(p) \in \mathcal{R}_1^{(\geq)}\,,\, s \geq 0\right\}\,. \tag{50}$$

According to the above remarks, it is now easy to realize that $\mathcal{R}$ and $\mathcal{R}^{(\geq)}$ are respectively homeomorphic to $\mathcal{R}_1 \times I\!R_+$, and $\mathcal{R}_1^{(\geq)} \times \overline{I\!R_+}$, so that we can conclude that:

1. $\mathcal{R}$ is $(q-1)$-dimensional and connected iff $\mathcal{R}_1$ is $(q-2)$-dimensional and connected;
2. $\overline{\mathcal{R}} \;=\; \mathcal{R}^{\geq}$ iff $\overline{\mathcal{R}}_1 \;=\; \mathcal{R}_1^{\geq}$. $\square$

The precise correspondence between $P\widehat{P}M$ and $A\widehat{P}M$ and between formal $A\widehat{P}M$ and $\widehat{P}$-matrices originating from CLG's has not yet been fully clarified. The following facts have however been proved in [11, 12]:

i) the $P\widehat{P}M$'s of r-type (q,q) are necessarily $A\widehat{P}M$'s of r-type (q,q);

ii) For $q \leq 4$, $P\widehat{P}M$'s of r-type (q,q) have been shown to be ≥ 0 only on a connected q-dimensional semi-algebraic subset of $\mathcal{Z} = I\!R^q$ and, for most of them, the boundary conditions have been checked. If this result could be shown to hold in general, there would be identity between $A\widehat{P}M$'s and $P\widehat{P}M$'s of r-type (q,q).

iii) For each choice of the degrees $\{d_1, d_2, \ldots, d_q\}$, there exists only a finite (or null) number of non equivalent $\widehat{P}$-matrices of r-type (q,q), at least for $q \leq 4$. This implies that the possible sets of degrees are limited by *selection rules*. All of these $P\widehat{P}M$'s can be organized in *towers* and the degrees of the elements of the same tower can be written in the form $d_\alpha = s d_\alpha^{(0)}$, $\alpha = 1, \ldots, q-1$, where s is a positive integer scale parameter. All the $\widehat{P}$-matrices of the same tower coincide for $p_q = 1$.

iv) Any $\widehat{P}$-matrix originating from a coregular CLG with no fixed points is necessarily equivalent to an $A\widehat{P}M$ of r-type (q,q). As a consequence, the selection rules on the sets of allowable degrees mentioned under item iii) hold true for all coregular CLG's. At present we do not know whether the converse holds also true, i.e., if every $A\widehat{P}M$ of r-type (q,q) is generated by a coregular CLG with no fixed points. A partial answer to this question has however been given in [18] and [19], where it has been checked that the $\widehat{P}$-matrices originating from all the finite coregular groups (which are the groups generated by reflections [20, 21]) and from all the coregular representations of compact simple Lie groups, with less than 5 basic invariants, can be found among the $A\widehat{P}M$'s listed in [11, 12].

The correspondence between $P\widehat{P}M$'s and $A\widehat{P}M$'s of r-type $(q,q-1)$ is much more difficult to study, for general values of q. Here we shall limit ourselves to note that, if $\widehat{P}(p)$ is a $\widehat{P}$-matrix of a CLG of r-type $(q,q-1)$ and $\widehat{F}(p) = 0$ is the (basic) relation among the elements of the MIB $\{p\}$, then, in F-bases $\{p'\}$, the couple $(\widehat{P}'(p'), \widehat{F}'(p'))$ is necessarily a *proper* solution of r-type $(q,q-1)$ of the canonical equation.

In the following section we shall start this analysis by determining all the proper $\widehat{P}$-matrices of r-type $(q,q-1)$, in the simplest case $q = 3$.

5 Non-coregular groups of r-type $(3,2)$

In this section we shall determine the $\widehat{P}$-matrices (and therefore the isotropy classes of the orbit spaces) of all the non-coregular groups of r-type $(3,2)$, i.e., with 3 basic polynomial invariants connected by only one independent relation. We shall start by determining the $P\widehat{P}M$'s of r-type $(3,2)$; from these we shall select the allowable ones.

5.1 Proper and allowable $\widehat{P}$-matrices of r-type $(3,2)$

We shall limit ourselves to sketch the procedure we have followed to determine all the $P\widehat{P}M$'s of r-type $(3,2)$. The most general form allowed by the conditions listed in §2.3 under **P1** and **P2** for the elements of the matrix $\widehat{P}(p)$ and by the weights of the $L_a(p)$ is the following:

$$
\begin{array}{rcl}
\widehat{P}_{11}(p) & = & d_1^2[p_1 a_1(p_2,p_3) + a_2(p_2,p_3)] \\
\widehat{P}_{12}(p) & = & d_1 d_2[p_1 b_1(p_3) + a_3(p_2,p_3)] \\
\widehat{P}_{22}(p) & = & d_2^2[p_1 b_2(p_3) + p_2 b_3(p_3) + b_4(p_3)] \\
\widehat{P}_{a3}(p) & = & 2d_a p_a \\
L_1(p) & = & d_1 a_4(p_2,p_3) \\
L_2(p) & = & d_2 b_5(p_3)\,.
\end{array}
\tag{51}
$$

where the a's and b's are unknown polynomial functions whose weights, determined according to (15) and (44) are specified in Table 1. The factorization of the d_i's is suggested by the structure of the equations we shall have to solve.

polynomial	weight	polynomial	weight	polynomial	weight
a_1	d_1-2	a_2	$2d_1-2$	a_3	d_1+d_2-2
a_4	d_1-2	b_1	d_2-2	b_2	$2d_2-d_1-2$
b_3	d_2-2	b_4	$2d_2-2$	b_5	d_2-2

Table 1: Weights of the unknown polynomials entering in the definition of $\widehat{P}(p)$

Since $\widehat{F}(p)$ has been required to be an irreducible (on the complex numbers) polynomial in the indeterminates $p_1,\ldots,p_q$, its gradient $\partial\widehat{F}(p)$ cannot vanish identically on the surface

$$\mathcal{Z} = \left\{p \in I\!R^q \ \mid\ \widehat{F}(p) \;=\; 0\right\}\,. \tag{52}$$

Therefore, thinking of the canonical equation as a system of linear equations, it is easy to realize that the determinant of the matrix $\widehat{P}(p)$ of the coefficients has to vanish where $\widehat{F}(p)$ vanishes. This means that $\widehat{F}(p)$ is necessarily a factor of $\det\widehat{P}(p)$ and, consequently:

$$w(F) \leq w(\det\widehat{P}) = \sum_1^3{}_a(2d_a-2). \tag{53}$$

The most general form allowed for $\widehat{F}(p)$ is therefore the following:

$$\widehat{F}(p) = f_3(p_3)\,p_1^3 + f_2(p_2,p_3)\,p_1^2 + f_1(p_2,p_3)\,p_1 + f_0(p_2,p_3), \tag{54}$$

where the f's are w-homogeneous polynomials of weights

$$w(f_j) \;=\; w(F) - j d_1, \qquad j = 0,\ldots,3 \tag{55}$$

and the condition $\partial\widehat{F}(0) = 0$ requires:

$$f_1(0,0) \;=\; 0, \qquad \partial f_0(0,0) \;=\; 0\,. \tag{56}$$

The overall normalization of $\widehat{F}(p)$ can be fixed arbitrarily.

The couple $(\widehat{P}(p), \widehat{F}(p))$ has to satisfy the canonical equation (27) and the additional conditions (43), (45) and (46). The derivative $\partial_3\widehat{F}(p)$ can be eliminated from these equations making use of the $w-$homogeneity condition on $\widehat{F}(p)$:

$$\sum_1^2{}_\alpha d_\alpha p_\alpha \partial_\alpha \widehat{F}(p) + 2\,p_3\partial_3\widehat{F}(p) = w(F)\,\widehat{F}(p)\,. \tag{57}$$

In this way one obtains for the canonical equation (27):

$$\sum_1^2{}_\beta \left(p_3\widehat{P}_{\alpha\beta}(p) - d_\alpha d_\beta p_\alpha p_\beta\right)\partial_\beta\widehat{F}(p) + w(F) d_\alpha p_\alpha \widehat{F}(p) \;=\; 0\,, \qquad \alpha = 1,2 \tag{58}$$

and for the additional condition (43):

$$\sum_1^2{}_\alpha \left(p_3 L_\alpha(p) - D d_\alpha p_\alpha\right)\partial_\alpha\widehat{F}(p) + D w(F)\widehat{F}(p) \;=\; 0\,. \tag{59}$$

The solution of (58) and (59), fulfilling the condition $\partial\widehat{F}(0) = 0$, can be obtained through the following steps:

1. Since in (58) and (59) there are no derivatives with respect to p_3, and the equations are w-homogeneous, it is advantageous to solve them first for $p_3 = 1$ and to reintroduce the dependence on p_3 in the solutions at the end. Then, let us set:

$$f_i(p_2, 1) \;=\; f_i(p_2),\;\; i = 0,1,2; \qquad f_3(1) \;=\; f_3; \tag{60}$$

$$a_i(p_2, 1) \;=\; a_i(p_2),\;\; i = 1,\ldots,4; \qquad b_j(1) \;=\; b_j,\;\; j = 1,\ldots,5 \tag{61}$$

2. The dependence on p_1 is made explicit after substituting in (58) and (59) the expressions (51) and (54) for $\widehat{P}_{ab}(p)$ and $\widehat{F}(p)$. Therefore, the principle of identity for polynomials allows to eliminate easily the variable p_1. One obtains in this way the following system of coupled algebro-differential equations, where w stands for $w(F)$:

$$(w - 3d_1)f_3 \;=\; 0\,,$$

$$3d_1 f_3 a_1(p_2) + (w - 2d_1) f_2(p_2) + d_2(b_1 - p_2) f_2'(p_2) \;=\; 0\,,$$

$$3d_1 f_3 a_2(p_2) + 2d_1 a_1(p_2) f_2(p_2) + (w - d_1) f_1(p_2) + d_2 a_3(p_2) f_2'(p_2) + d_2(b_1 - p_2) f_1'(p_2) \;=\; 0\,,$$

$$2d_1a_2(p_2)f_2(p_2) + d_1a_1(p_2)f_1(p_2) + wf_0(p_2) + d_2a_3(p_2)f_1'(p_2) + d_2(b_1 - p_2)f_0'(p_2) \; = \; 0\,,$$

$$d_1a_2(p_2)f_1(p_2) + d_2a_3(p_2)f_0'(p_2) \; = \; 0\,,$$

$$(3b_1d_1 + wp_2 - 3d_1p_2)f_3 + b_2d_2f_2'(p_2) \; = \; 0\,,$$

$$3d_1f_3a_3(p_2) + (2b_1d_1 + wp_2 - 2d_1p_2)f_2(p_2) + d_2(b_4 + b_3p_2 - p_2^2)f_2'(p_2) + b_2d_2f_1'(p_2) \; = \; 0\,,$$

$$2d_1a_3(p_2)f_2(p_2) + (b_1d_1 + wp_2 - d_1p_2)f_1(p_2) + d_2(b_4 + b_3p_2 - p_2^2)f_1'(p_2) + b_2d_2f_0'(p_2) \; = \; 0\,,$$

$$d_1a_3(p_2)f_1(p_2) + wp_2f_0(p_2) + d_2(b_4 + b_3p_2 - p_2^2)f_0'(p_2) \; = \; 0\,,$$

$$D(w - 3d_1)f_3 \; = \; 0\,,$$

$$3d_1f_3a_4(p_2) + D(w - 2d_1)f_2(p_2) + d_2(b_5 - Dp_2)f_2'(p_2) \; = \; 0\,,$$

$$d_1a_4(p_2)f_2(p_2) + D(w - d_1)f_1(p_2) + d_2(b_5 - Dp_2)f_1'(p_2) \; = \; 0\,,$$

$$d_1a_4(p_2)f_1(p_2) + Dwf_0(p_2) + d_2(b_5 - Dp_2)f_0'(p_2) \; = \; 0\,.$$

3. The solution of the system of equations just written is much more lengthy and trickier. In principle, it could be reduced to the solution of a system of algebraic equations by expanding the unknown polynomial functions in powers of p_2 and identifying to 0 the coefficients of homonymous powers of p_2 in each equation. It has to be recalled however, that the weights of the polynomials are functions of the degrees d_i, which are parameters. So, the high number of variables one is obliged to introduce in this way and the high number of coupled algebraic equations to solve make this standard procedure quite difficult to handle.

   The easiest way to obtain the solutions seems to be through a combined use of algebraic and integro–differential methods, with a clever choice of the order in which to solve the various equations.

It would be too long to describe the details of the calculations that led to the determination of all the solutions of the system of equations (58), (59) and (56). We shall limit ourselves to report the solutions, discarding those in which $\widehat{F}(p)$ turns out to be reducible on the complexes. The proper solutions will be collected in 3 families, S1, S2 and S3, corresponding respectively to the degrees $(d_1, d_2) = (k(2m+1), 2k)$, $(d_1, d_2) = (6k, 4k)$, and $(d_1, d_2) = (k+1, k+1)$, where m and k are positive integers. Each family will be discussed separately in each of the following three subsections. For each family we shall determine the number of

distinct equivalence classes of $\widehat{P}$-matrices picking up for each class a representative $\widehat{P}$-matrix, chosen so that the numerical coefficients of all the polynomial involved are integer numbers. The allowability conditions will finally be checked for each representative $\widehat{P}$-matrix.

For all the solutions, the overall normalization of $\widehat{F}(p)$ will be chosen so that the coefficient of the highest power of p_1, which turns out to be always a constant, equals 1.

5.1.1 Solution S1

The family S1 of proper solutions is found in correspondence with the degrees

$$d_1 = k(1+2m)\,, \qquad d_2 = 2k\,, \qquad k, m \in \mathbb{N}_*\,. \tag{62}$$

In the rest of this section, k and m will be considered as fixed. For the unknown polynomial functions $L_1, L_2, \widehat{F}, \widehat{P}_{ij},\ i,j = 1,2$ one finds the following expressions, in terms of two real parameters, $b_3 \neq 0$ (defined in (51)) and c (originating as an integration constant):

$$L_1 \ = \ 0; \qquad L_2(p_3) \ = \ d_2 D b_3\, p_3^{k-1}\,. \tag{63}$$

$$\widehat{F} = p_1^2 + c(p_2 - b_3 p_3^k)^{2m+1}, \tag{64}$$

$$\begin{aligned} \widehat{P}_{11}(p) &= d_1^2\, b_3\, c\, p_3^{k-1} \left(p_2 - b_3 p_3^k\right)^{2m} \\ \widehat{P}_{12}(p) &= 0 \\ \widehat{P}_{22}(p) &= d_2^2\, b_3\, p_2\, p_3^{k-1} \end{aligned} \tag{65}$$

Different values of the parameters b_3 and c do not necessarily determine non-equivalent $\widehat{P}$-matrices. In fact,

Proposition 5.1 *For each fixed choice of (m, k), the $\widehat{P}$-matrices determined by the family S1 of proper solutions form two distinct classes of equivalent $\widehat{P}$-matrices.*

Proof: By means of the following formal MIBT

$$\begin{aligned} p_1' &= \left|b_3^{2m+1} c\right|^{-\frac{1}{2}} p_1 \\ p_2' &= b_3^{-1}\, p_2 \\ p_3' &= p_3\,; \end{aligned} \tag{66}$$

after setting

$$\epsilon = \mathrm{sign}(b_3 c),$$

one obtains from (18), (64), (65) and (62):

$$\widehat{F}'(p') = {p'_1}^2 + \epsilon \left(p'_2 - {p'_3}^k\right)^{2m+1} ,$$

$$\begin{aligned}
\widehat{P}'_{11}(p') &= d_1^2\, \epsilon\, {p'_3}^{k-1} \left(p'_2 - {p'_3}^k\right)^{2m} \\
\widehat{P}'_{12}(p') &= 0 \\
\widehat{P}'_{22}(p') &= d_2^2\, p'_2\, {p'_3}^{k-1},
\end{aligned} \tag{67}$$

and

$$\det(\widehat{P}'(p')) = -4\, d_1^2\, d_2^2\, p'_2 {p'_3}^{k-1} \left\{ {p'_1}^2 - \epsilon({p'_3}^k - p'_2)^{2m+1} \right\} .$$

In this way all the arbitrary parameters, but for sign$(b_3 c)$, have been absorbed in the redefinition of the MIB. It is now evident that no MIBT can cause, as unique consequence, a change of sign in front of ϵ in (67).

For each fixed choice of (k, m), there are therefore two distinct classes of equivalent $\widehat{P}$-matrices, represented by the $\widehat{P}$-matrices defined in (67), respectively for $\epsilon = 1$ and $\epsilon = -1$. $\square$

Proposition 5.2 *The proper solutions of the family S1 do not determine allowable $\widehat{P}$-matrices.*

Proof: It will be sufficient to check the allowability conditions on the representative $\widehat{P}$-matrices defined in (67). To this end, let us determine the semi-positivity domain of $\widehat{P}'(p')$ in $\mathcal{Z}$, separately for $\epsilon = \pm 1$.

For $\epsilon = -1$ and $p'_3 = 1$, from Theorem 4.1 and Lemma 4.1 we immediately realize that $\widehat{P}'(p') \geq 0$ only at $p'_2 = 1$, $p'_1 = 0$, which is a 0-dimensional set. Therefore, for $\epsilon = -1$, the matrix $\widehat{P}'(p')$ is not allowable.

For $\epsilon = +1$, the subset $\mathcal{R}_1^{(\geq)} \subset \Pi \cap \mathcal{Z}$ where $\widehat{P}'(p') \geq 0$ is determined by the following conditions:

$$\mathcal{R}_1^{(\geq)} = \Pi \cap \mathcal{R}^{(\geq)} = \left\{ (p'_1, p'_2) \;\mid\; {p'_1}^2 = (1 - p'_2)^{2m+1}, \qquad 0 \leq p'_2 \leq 1 \right\} , \tag{68}$$

where, obviously, rank$\,(\widehat{P}'(p')) = 2$ in the interior of $\mathcal{R}_1^{(\geq)}$ and rank$\,(\widehat{P}'(p')) = 1$ on the boundary. The variety $\mathcal{Z}$ admits the following parametric representation:

$$\begin{aligned}
p'_1 &= t^{2m+1} \\
p'_2 &= 1 - t^2
\end{aligned} \tag{69}$$

and $\mathcal{R}_1^{(\geq)}$ corresponds to values of $t \in [-1, 1]$, the singular strata corresponding to $t = 0, \pm 1$.

stratum	t	defining relations	generators	boundary conditions
$\sigma^{(1)}$	0	$p'_1 = 0$ $p'_2 = {p'_3}^k$	$f^{(1)} = p'_1$ $f^{(2)} = p'_2 - {p'_3}^k$	satisfied
$\sigma^{(2)}$	1	${p'_1}^2 = {p'_3}^{k(2m+1)}$ $p'_2 = 0$ $p'_1 > 0$	$f^{(1)} = {p'_1}^2 - {p'_3}^{k(2m+1)}$ $f^{(2)} = p'_2$	not satisfied
$\sigma^{(3)}$	-1	${p'_1}^2 = {p'_3}^{k(2m+1)}$ $p'_2 = 0$ $p'_1 < 0$	$f^{(1)} = {p'_1}^2 - {p'_3}^{k(2m+1)}$ $f^{(2)} = p'_2$	not satisfied

Table 2: Defining relations for the singular primary strata, generators and boundary conditions of solution S1 in the case $\epsilon = +1$.

The set of points $\mathcal{R}_1$ where $\widehat{P}'(p') \geq 0$ and rank $(\widehat{P}(p)) = 2$ is not connected, as it is evident from Fig. (1). We conclude therefore that, owing to Lemma 4.1, the two distinct classes of equivalent $\widehat{P}$-matrices obtained from the family S1 of proper solutions are *not allowable.* $\square$

We have reported in Table 2 the equations defining the singular primary strata $\sigma^{(i)}$ of $\mathcal{R}^{(\geq)}$ and a possible choice for the generators of the associated ideals $\mathcal{I}(\sigma^{(i)})$.

It is also easy to check that the boundary conditions are not satisfied at the singular primary strata $\sigma^{(2)}$ and $\sigma^{(3)}$ defined in Table 2.

5.1.2 Solution S2

The solution S2 is found in correspondence with the degrees

$$d_1 = 6k, \qquad d_2 = 4k, \qquad k \in \mathbb{N}_*$$

and is defined by the following expressions of the unknown polynomial functions:

$$L_1(p_2, p_3) = \frac{d_1 D b_1}{4b_2} \left[2p_2 + (3b_1 + 2b_3)p_3^{2k}\right] p_3^{k-1}; \qquad L_2(p_3) = \frac{d_2 D}{2}(3b_1 + 2b_3)p_3^{2k-1}; \tag{70}$$

$$\begin{aligned}\widehat{F}(p) = \frac{1}{8b_2^2} \Big\{ & 8b_2^2 p_1^2 + 4b_1 b_2 p_1 p_3^k[(3b_1 + 2b_3)p_3^{2k} - 6p_2)] + b_1[4p_2^3 - 12b_3 p_2^2 p_3^{2k} + \\ & (9b_1^2 + 24b_1 b_3 + 12b_3^2)p_2 p_3^{4k} - (9b_1^3 + 21b_1^2 b_3 + 16b_1 b_3^2 + 4b_3^3)p_3^{6k}\Big\},\end{aligned} \tag{71}$$

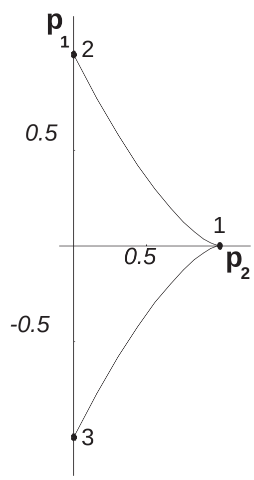

Figure 1: Stratification of $\mathcal{R}_1^{(\geq)}$ for solution S1. The numbers j, $j = 1, 2, 3$, label the singular primary strata $\sigma^{(j)}$ defined in Table 2.

$$
\begin{aligned}
\widehat{P}_{11}(p) &= \frac{b_1 d_1^2}{16 b_2^2}\, p_3^{k-1} \Big\{ 4b_2 p_1 \Big[2p_2 + (b_1 - 2b_3) p_3^{2k} \Big] + 4(b_1 + 2b_3) p_2^2 p_3^k - \\
&\quad 8b_3(b_1 + 2b_3) p_2 p_3^{3k} + (3b_1^3 + 14 b_1^2 b_3 + 20 b_1 b_3^2 + 8 b_3^3) p_3^{5k} \Big\} \\
\widehat{P}_{12}(p) &= \frac{b_1 d_1 d_2}{8 b_2}\, p_3^{2k-1} \Big\{ 8b_2 p_1 + 6(b_1 + 2b_3) p_2 p_3^k - (3b_1^2 + 8 b_1 b_3 + 4 b_3^2) p_3^{3k} \Big\} \\
\widehat{P}_{22}(p) &= \frac{d_2^2}{4}\, p_3^{k-1} \Big\{ 4b_2 p_1 + 4b_3 p_2 p_3^k + (3b_1^2 + 2b_1 b_3) p_3^{3k} \Big\} ;
\end{aligned}
\tag{72}
$$

the determinant of $\widehat{P}(p)$ is

$$
\begin{aligned}
\det(\widehat{P}(p)) &= \frac{d_1^2 d_2^2}{4 b_2^2}\, p_3^{k-1} \Big(2b_2 p_1 + b_1 p_2 p_3^k + 2b_3 p_2 p_3^k \Big) \{ -8 b_2^2 p_1^2 + 4 b_1 b_2 [6p_2 - \\
&\quad (3b_1 + 2b_3) p_3^{2k} \Big] p_1 p_3^k + b_1 \Big(-4p_2^3 + 12 b_3 p_2^2 p_3^{2k} - 3(3b_1^2 + 8b_1 b_3 + \\
&\quad 4b_3^2) p_2 p_3^{4k} + (9b_1^3 + 21 b_1^2 b_3 + 16 b_1 b_3^2 + 4 b_3^3) p_3^{6k} \Big) \Big\} .
\end{aligned}
\tag{73}
$$

For each $k \in \mathbb{N}_*$, the solution depends on the real parameters b_1, b_2 and b_3, whose values are restricted by the following condition:

$$b_1\, b_2 \neq 0\,. \tag{74}$$

In the rest of this section k will be considered as fixed.

Proposition 5.3 *For each k, the $\widehat{P}$-matrices determined by the family S2 of proper solutions form a one-parameter collection of distinct classes of equivalent $\widehat{P}$-matrices.*

Proof: It will be advantageous to write the matrix $\widehat{P}(p)$ defined in (72) in a different (non-F-) basis.

With the following formal MIBT:

$$\begin{aligned} p'_1 &= \frac{1}{b_1^2}\left[p_1 + b_1(3b_1+2b_3)p_3^{3k} - 6b_1p_2p_3^k\right] \\ p'_2 &= \frac{1}{b_1}\left[2p_2 - (3b_1+2b_3)p_3^{2k}\right] \\ p'_3 &= p_3\,, \end{aligned} \tag{75}$$

after setting

$$(3b_1+2b_3)/b_1 = z \tag{76}$$

one obtains:

$$\widehat{F}'(p') \;=\; {p'_1}^2 + {p'_2}^3\,, \tag{77}$$

$$\begin{aligned} \widehat{P}'_{11}(p') &= {d_1}^2 p'_2 {p'_3}^{k-1}(-p'_1 + zp'_2{p'_3}^k) \\ \widehat{P}'_{12}(p') &= -d_1d_2{p'_3}^{k-1}(zp'_1{p'_3}^k + {p'_2}^2) \\ \widehat{P}'_{22}(p') &= {d_2}^2{p'_3}^{k-1}(p'_1 - zp'_2{p'_3}^k), \end{aligned} \tag{78}$$

and

$$\det(\widehat{P}'(p')) \;=\; -4{d_1}^2{d_2}^2{p'_3}^{k-1}\left(z^2{p'_3}^{3k} + p'_1 + (1+z)p'_2{p'_3}^k\right)({p'_1}^2 + {p'_2}^3)\,. \tag{79}$$

We are left therefore with a unique free parameter z. A direct check shows that $\widehat{P}$-matrices corresponding to different values of z cannot be related by MIBT's; the parameter z labels, therefore, the elements of a one-parameter collection of non-equivalent $\widehat{P}$-matrices, each representing a class of equivalent $\widehat{P}$-matrices. □

Let us now prove that

Proposition 5.4 *The proper solutions of the family S2 determine only one class of equivalent $A\widehat{P}M$'s. A representative $\widehat{P}$-matrix is defined by (78) for $z = 0$.*

stratum	t	defining equations	generators	boundary conditions
$\sigma^{(1)}$	1	$p'_1 = {p'_3}^{3k}$ $p'_2 = -{p'_3}^{2k}$	$f^{(1)} = p'_1 - {p'_3}^{3k}$ $f^{(2)} = p'_2 + {p'_3}^{2k}$	satisfied
$\sigma^{(2)}$	0	$p'_1 = 0$ $p'_2 = 0$	$f^{(1)} = p'_1$ $f^{(2)} = p'_2$	satisfied

Table 3: Defining equations for the primary strata, generators and boundary conditions of solution S2 for $z = 0$.

Proof: It will be sufficient to analyse the representative matrices defined in (78).

The algebraic variety $\mathcal{Z}$ determined by the relation $\widehat{F}'(p')$ can be characterized by means of the following parametric equations:

$$\begin{aligned} p'_1 &= t^3 \\ p'_2 &= -t^2 \,. \end{aligned} \tag{80}$$

For each fixed value of z, let us determine the semi-positivity domain $\mathcal{R}_1^{(\geq)}$ of $\widehat{P}'(p')$ in $\Pi \cap \mathcal{Z}$.

An immediate application of Theorem 4.1 shows that the region $\mathcal{R}_1$ where $\widehat{P}'(t^3, -t^2, 1) \geq 0$ and has rank 2 is determined by the condition:

$$t^2(z + t - t^2) > 0 \,, \tag{81}$$

while the region where $\widehat{P}'(t^3, -t^2, 1) \geq 0$ and has rank 1 is determined by the condition

$$t = 0 \,, \qquad \text{or} \qquad t = \frac{1}{2}\left(1 \pm \sqrt{1 + 4z}\right) \,. \tag{82}$$

Therefore, condition *i)* of Lemma 4.1 is satisfied if and only if $z = 0$. For $z = 0$:

$$\mathcal{R}_1^{(\geq)} = \left\{ (p'_1, p'_2) \mid p'_1 = t^3,\, p'_2 = -t^2,\ 0 \leq t \leq 1 \right\} .$$

In order to check the boundary conditions at the singular strata (item *ii)* of Lemma 4.1) in the case $z = 0$, in Table 3 we have reported the equations defining the singular primary strata $\sigma^{(i)}$ of $\mathcal{R}^{(\geq)}$ and a possible choice for the generators of the associated ideals $\mathcal{I}(\sigma^{(i)})$. At this point it is easy to check that the boundary conditions are satisfied at all the singular primary strata, denoted by $\sigma^{(1)}$ and $\sigma^{(2)}$ in Fig. 2. □

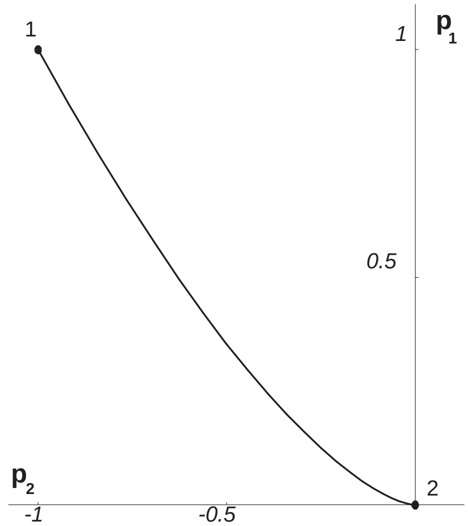

Figure 2: Stratification of $\mathcal{R}_1^{(\geq)}$ for solution S2 in the case $z = 0$. The numbers j, $j = 1, 2$, label the singular primary strata $\sigma^{(j)}$ defined in Table 3.

5.1.3 Solution S3

The family S3 of proper solutions is found in correspondence with the degrees

$$d_1 = d_2 = k \in \mathbb{N}, \quad k \geq 2 \tag{83}$$

and for

$$D = 0; \qquad n = 2. \tag{84}$$

In the rest of this section, k will be considered as fixed.

For the unknown polynomial functions one finds the following expressions:

$$L_1 = 0, \qquad L_2 = 0;$$

$$\widehat{F}(p) = \frac{1}{4c_2}\left(-4c_1c_2{p_3}^k + 4c_2p_1^2 + 4c_2c_3p_1p_2 + 4c_1p_2^2 + c_2c_3^2p_2^2\right); \tag{85}$$

$$\begin{aligned} \widehat{P}_{11}(p) &= \frac{{d_1}^2}{4}(4c_1 + c_2c_3^2)p_3^{k-1} \\ \widehat{P}_{12}(p) &= -\frac{d_1d_2}{2}c_2c_3p_3^{k-1} \\ \widehat{P}_{22}(p) &= {d_2}^2c_2p_3^{k-1}, \end{aligned} \tag{86}$$

where $c_1,\ c_2$ and c_3 are real parameters, satisfying the condition:

$$c_1\, c_2 \;\neq\; 0\,. \tag{87}$$

Proposition 5.5 *For each fixed k, the $\widehat{P}$-matrices determined by the family S3 of proper solutions form four distinct classes of equivalent $\widehat{P}$-matrices.*

Proof: The parameters c_3, $|c_1|$ and $|c_2|$ entering the definition of $\widehat{P}(p)$ in (86) can be eliminated by means of the following formal MIBT:

$$\begin{aligned} p'_1 &= |c_1|^{-1/2}\,(p_1 + c_3 p_2/2) \\ p'_2 &= |c_2|^{-1/2} p_2 \\ p'_3 &= p_3\,. \end{aligned} \tag{88}$$

After setting

$$\epsilon_i = \mathrm{sign}(c_i)\,, \qquad i = 1, 2\,,$$

in the new F-basis the non fixed elements of the $\widehat{P}$-matrix defined in (86) assume the following simple form:

$$\begin{aligned} \widehat{P}'_{11}(p') &= {d_1}^2 \epsilon_1 {p'_3}^{k-1} \\ \widehat{P}'_{12}(p') &= 0 \\ \widehat{P}'_{22}(p') &= {d_2}^2 \epsilon_2 {p'_3}^{k-1}\,. \end{aligned} \tag{89}$$

From (89)

$$\det \widehat{P}'(p') = 4{d_1}^2{d_2}^2\, \epsilon_1\epsilon_2 \left({p'_3}^k - \epsilon_1 {p'_1}^2 - \epsilon_2 {p'_2}^2\right) \;=\; -\epsilon_2\, \widehat{F}(p'). \tag{90}$$

It is now trivial to realize that the 4 matrices defined in (89) for $\epsilon_1 = \pm 1$ and $\epsilon_2 = \pm 1$ cannot be related by a MIBT. For each fixed k, we are left therefore with only 4 distinct classes of equivalent $\widehat{P}$-matrices. □

Proposition 5.6 *The proper solutions of the family S3 determine only one class of equivalent $A\widehat{P}M$'s. A representative $\widehat{P}$-matrix is defined by (89) for $\epsilon_1 = \epsilon_2 = 1$.*

Proof: From (89) and (90) it is easy to realize that the semipositivity domain $\mathcal{R}_1^{(\geq)}$ of $\widehat{P}'(p')$ in $\Pi \cap \mathcal{Z}$ is a 1-dimensional connected semi-algebraic set only if $\epsilon_1 = \epsilon_2 = 1$. In this case, $\mathcal{R}^{(\geq)}$ is the surface of the upper half of a cone, and $\mathcal{R}_1^{(\geq)}$ is the unit circle (see [8], Fig. 6, pag. 332).

It is trivial to check that the conditions listed in Lemma 4.1 are satisfied for $\epsilon_1 = \epsilon_2 = 1$. Moreover, since the unique singular stratum is the origin of $\mathbb{R}^q$, the boundary conditions are

certainly satisfied. We conclude therefore that only for $\epsilon_1 = \epsilon_2 = 1$ the $P\widehat{P}M$'s defined in (89) are *allowable.* $\square$

An interesting fact to note is that the $A\widehat{P}M$'s we have found have the same form as the $A\widehat{P}M$'s of r-type $(3,3)$ of the solution of class $I(1,1)$ reported in [11].

5.2 Generating groups

As stressed in §3 the $\widehat{P}$-matrix generated by a CLG of r-type $(3,2)$ must be equivalent to an A$\widehat{P}$M of r-type (3,2). The results reported in the previous section show that it must be equivalent to a matrix in one of the families S2 or S3. As just pointed out, we cannot be sure, a priori, that the converse holds true too, *i.e.,* that each of the matrices in the families S2 and S3 are necessarily generated by a CLG. In the next two subsections we shall study the problem in greater detail, separately for the solutions of the families S2 and S3. The analysis will lead us to discover that a further selection has to be done.

5.2.1 Allowable solutions of the family S2

For each of the solutions of the family S2 the existence of a generating group can be excluded on the basis of the following proposition:

Proposition 5.7 If $f_1(p)$ and $f_2(p)$ are w-homogeneous prime polynomials and $\widehat{F}(p) = f_1(p)^{n_1} - f_2(p)^{n_2}$, $n_1, n_2 > 1$, $n_1, n_2 \in \mathbb{N}$, then, $\widehat{F}(p)$ cannot define a basic relation among the elements of a minimal integrity basis of a compact linear group of r-type $(q, q-1)$.

Proof: Let us assume, in fact, that $\{p_1(x), \ldots, p_q(x)\}$ is a MIB of a compact linear group of r-type $(q, q-1)$ and that the polynomial $\widehat{F}(p)$, defined in the statement, defines the basic relation. Then

$$F(p(x)) \;=\; f_1(p(x))^{n_1} - f_2(p(x))^{n_2} \;=\; 0 \tag{91}$$

is an identity in x and should be the lowest degree relation among the p_a's. Let us assume, without loss of generality, that $n_2 \geq n_1(>1)$. Then, Eq. (91) shows that $[f_1(p(x))/f_2(p(x))]^{n_1}$, and consequently $f_1(p(x))/f_2(p(x))$, is to be a G-invariant homogeneous polynomial $h(x)$:

$$f_1(p(x)) \;=\; h(x) f_2(p(x)). \tag{92}$$

Now, the degree of $h(x)$ is certainly lower than the degree of $F(p(x))$, therefore there is a unique w-homogeneous polynomial $\hat{h}(p)$ such that $\hat{h}(p(x)) = h(x)$, for all $x \in \mathbb{R}^n$ and

$$\Big(f_1(p) - \hat{h}(p) f_2(p)\Big)\Big|_{\mathcal{Z}} \;=\; 0\,. \tag{93}$$

Since the weight of the polynomial $f_1(p) - \hat{h}(p) f_2(p)$ is lower than the weight of $F(p)$, equation (93) can be extended to the whole of $\mathbb{R}^q$. This contradicts the assumption that $f_1(p)$ and $f_2(p)$ are prime polynomials. $\square$

Proposition 5.7 suggests the following model for the generation of proper solutions of class S2.

Let $q(x) = (q_1(x), q_2(x))$ define a MIB of a CLG, with degrees $(2k, 2)$. It is not restrictive to assume that the associated $\widehat{P}$-matrix has the following form [10]:

$$\widehat{Q}(q_1, q_2) \;=\; \begin{pmatrix} k^2 q_2^{k-1} & 2kq_1 \\ 2kq_1 & 4q_2 \end{pmatrix}. \tag{94}$$

Let us now define the following non-regular set $\{p\}$ of homogeneous polynomial invariants:

$$p_\alpha(x) \;=\; f_\alpha(q(x)), \quad \alpha = 1,2; \qquad p_3(x) \;=\; q_2(x), \tag{95}$$

where the f_α's are w-homogeneous polynomials.

The non fixed elements of the matrix $P(x)$, associated according to (14) to the invariants $\{p_1(x), p_2(x), p_3(x)\}$, can be written in the following form:

$$P_{\alpha\beta}(x) \;=\; \langle\; \partial f_\alpha(q(x)) \;,\; \widehat{Q}(q(x)) \partial f_\beta(q(x)) \;\rangle, \qquad \alpha, \beta = 1,2. \tag{96}$$

For general $f_1(q)$ and $f_2(q)$, the set $\{p(x)\}$ is not an integrity basis and the $P_{\alpha\beta}(x)$ cannot be expressed in terms of polynomials in $p_1(x), p_2(x)$ and $p_3(x)$. When this is possible, i.e., when a matrix $\widehat{P}(p(x))$ exists such that $P(x) \;=\; \widehat{P}(p(x))$, then $\{p(x)\}$ will be called a *pseudo integrity basis* (abbreviated in PIB).

We shall limit ourselves to determine those PIB's which are relevant to the interpretation of the proper solutions of class S2.

A simple calculation shows that the following mono-parametric family of couples of w-homogeneous polynomials $(f_1(q_1, q_2), f_2(q_1, q_2))$ of weights $d_1 = 6k$, $d_2 = 4k$ give rise to PIB's $\{p\}$:

$$\begin{aligned} p_1 &= f_1(q_1, q_2) \;=\; q_1^3 - 3s^2 q_1 q_2^{2k} \\ p_2 &= f_2(q_1, q_2) \;=\; q_1^2 + 2s q_1 q_2^k \\ p_3 &= q_2. \end{aligned} \tag{97}$$

By eliminating q_1 and q_2 from the three equations in (97), the following relation among the p_a's is easily found:

$$\widehat{F}(p) = \frac{1}{16}\left(16p_1^2 - 16p_2^3 + 48sp_1p_2p_3^k + 24s^2p_2^2p_3^{2k} + 4s^3p_1p_3^{3k} + 3s^4p_2p_3^{4k}\right). \tag{98}$$

The $\widehat{P}$-matrix associated to the PIB $\{p\}$ has the following non fixed elements:

$$
\begin{aligned}
P_{11}(p) &= \frac{d_1^2}{16} p_3^{k-1} \left(-16 s^2 p_2^2 p_3^k + 32 s^3 p_1 p_3^{2k} + 24 s^4 p_2 p_3^{3k} + 16 p_2^2 p_3^k + \right. \\
&\quad \left. -32 s p_1 p_3^{2k} - 24 s^2 p_2 p_3^{3k} + s^4 p_3^{5k} \right) \\
P_{12}(p) &= \frac{d_1 d_2}{8} p_3^{k-1} \left(4 s p_2^2 - 12 s^2 p_1 p_3^k - 9 s^3 p_2 p_3^{2k} + 8 p_1 p_3^k + 4 s p_2 p_3^{2k} - s^3 p_3^{4k} \right) \\
P_{22}(p) &= \frac{d_2^2}{4} p_3^{k-1} \left(4 s p_1 + 3 s^2 p_2 p_3^k + 4 p_2 p_3^k + s^2 p_3^{3k} \right) .
\end{aligned} \tag{99}
$$

With the following MIBT:

$$
\begin{aligned}
p_1' &= \frac{1}{8s^3} \left(8 p_1 + 12 s p_2 p_3^k + s^3 p_3^{3k} \right) \\
p_2' &= -\frac{1}{4s^2} \left(4 p_2 + s^2 p_3^{2k} \right) \\
p_3' &= p_3 ,
\end{aligned} \tag{100}
$$

the matrix $\widehat{P}(p)$ is changed into the $\widehat{P}$-matrix of Eq. (78), provided that

$$
z = -\frac{s^2 - 4}{4s^2} .
$$

It would be easy to check that also the $P\widehat{P}M$'s of the family S1 can be generated in an analogous way.

We shall conclude this section with a comment. From a strictly rational point of view, the restriction on the possible form of the basic relation among the elements of a MIB of a group of r-type $(q, q-1)$ should have been included in the definition of $A\widehat{P}M$'s. We have preferred to introduce it a posteriori to stress the fact that the necessity of this further condition has not been suggested to us by known results in invariant theory, but by our $\widehat{P}$-matrix approach to the study of orbit spaces of CLG's.

5.2.2 Allowable $\widehat{P}$-matrices of the family S3

Contrary to what happens for the $A\widehat{P}M$'s of the family S2, it is not difficult to find a generating group of the $A\widehat{P}M$'s of the family S3. According to (84) it has to be searched for among the linear groups acting in 2-dimensional spaces.

Let us consider for instance the group $\mathbb{Z}_n$ generated by the following transformation of the complex plane:

$$
z' = \exp\left(-\imath \frac{2\pi}{n} \right) z , \qquad z = x_1 + \imath x_2 \in \mathbb{C} .
$$

It is then evident that a MIB for G is the following one:

$$
p_1 = \Im(z^n), \qquad p_2 = \Re(z^n) , \qquad p_3 = z z^* = x_1^2 + x_2^2 .
$$

The MIB is not regular, since its elements satisfy the following identity:

$$p_1(x)^2 \;=\; p_3(x)^n - p_2(x)^2,$$

which exactly corresponds to the solution $\widehat{F}(p)$ we have found in our approach. The image of the orbit space $\overline{\mathcal{S}_1}$ is the unit circle. The group $\mathbb{Z}_n$ provides a 2-dimensional representation of the point group C_n, $n \geq 2$, that is the cyclic group of rotations about an axis of the n-th order.

As already noted, the $A\widehat{P}M$'s of r-type $(3, 2)$ of the solution S3 are equivalent to the $A\widehat{P}M$'s of r-type $(3, 3)$ of class $I(1, 1)$ reported in [11]. Generating groups of the first element of the family are, for instance, the linear groups $\mathrm{SO}(n, \mathbb{R})$ acting in $\mathbb{R}^n \oplus \mathbb{R}^n$ for $n \geq 3$.

6 Concluding remarks

To conclude, we would like to stress the following more or less unexpected facts emerging from our analysis of non-coregular CLG's with only one relation among the elements of their MIB's (class $\mathcal{T}(3, 2)$):

- Coregular and non-coregular groups may share the same $\widehat{P}$-matrix.
- There is only one mono-parametric discrete family of allowable non-equivalent $\widehat{P}$-matrices $\widehat{P}^{(k)}(p)$, $\mathbb{N} \ni k \geq 2$, whose elements may be generated by groups $G \in \mathcal{T}(3, 2)$. The degrees of the p_a's are $d_1 = d_2 = k \geq 2$, $d_3 = 2$ and, with a convenient choice of the p_a's, the basic relation can be written in the form $\widehat{F}^{(k)}(p) \;=\; p_1^2 + p_2^2 - p_3^k$.
- Every allowable $\widehat{P}$-matrix of the family is generated by at least a group $G \in \mathcal{T}(3, 2)$.
- If the action of the groups is restricted to the unit sphere $S^{(n-1)}$ of $\mathbb{R}^n$ (which is not essentially restrictive for what concerns the characterization of the orbit space), all the orbit spaces $S^{(n-1)}/G$, $G \in \mathcal{T}(3, 2)$ turn out to be isomorphic.

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